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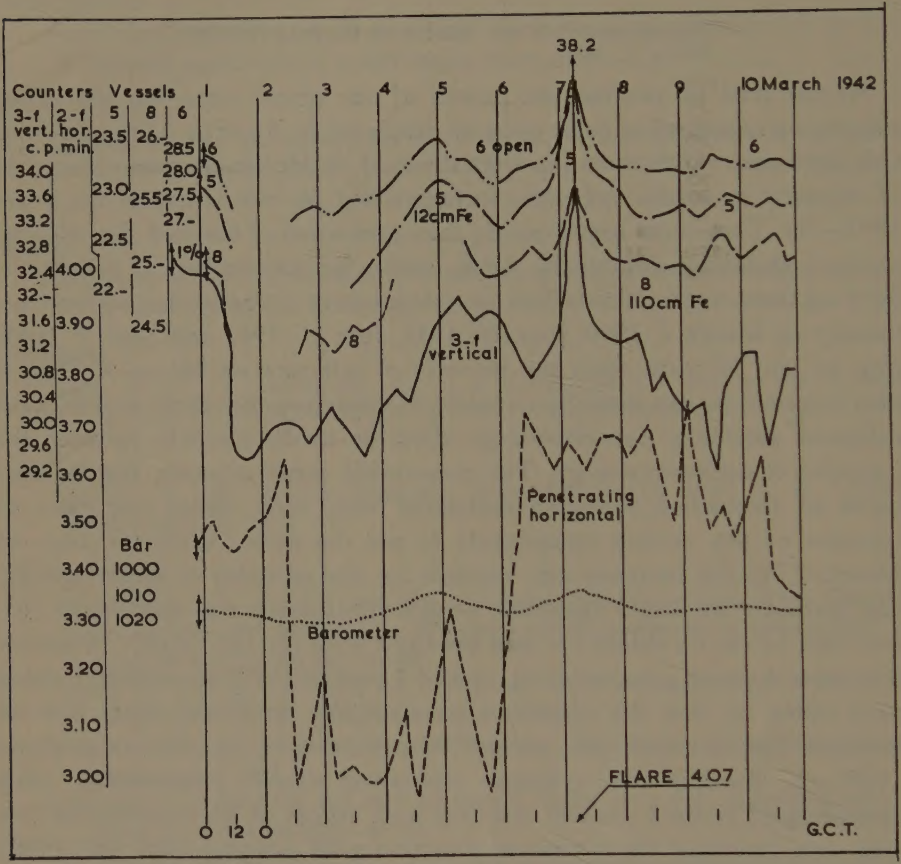
Physics. — *Solar flares and the origin of Cosmic Radiation.* III. By J. CLAY, H. F. JONGEN and A. J. DIJKER. (Natuurkundig Laboratorium, University of Amsterdam.)

(Communicated at the meeting of October 29, 1949.)

At the time of reading the proofs of our recent paper on the above subject we happened to come upon an article by A. EHMERT, Ultrastrahlung von der Sonne (Ultraradiation from the sun) [1]. Herein 28 cases of increase of intensity in cosmic radiation are signalized for the course of the years 1941—'43. They were found during the registration of twofold coincidences without shielding, corrections being made for metereological influences. In 4 of these cases correlations are stated with chromospheric eruptions, namely on March 7, 1942, June 20, 1941, July 11, 1941 and Sept. 2, 1941. Fig. 11 and 12 show, that the increase of intensity on March 7, 1942 is also recorded by our ionisation vessels. Considering that these vessels have different shielding, our recordings allow us to discriminate between the particles of different energy. The remarkable result appears, that in most cases of intensified radiation correlated with solar flares, the ratio of increase of the various components is not the same. With the flare of March 7 '42 the increase was smallest for the particles of larger energy. On that date we had 3 ionisation vessels functioning, one unshielded (6), one with 12 cm Fe shield (5) and the third with 110 cm Fe (8). Moreover 3 counter systems were working, one of 3 counters of 3 cm diameter above each other, so that the opening cone was 26°, no shield either over or between the counters; the amount of coincidences recorded was about 1900 p.h. Further two systems, registering double coincidences, each consisting of twice 3 parallel counters with 10 cm of Pb between the two sets, one arranged for a vertical direction with opening cone 48° (4800 c.p.h.), the other for a horizontal direction, leaving an opening of 24°. The latter one gave 200 c.p.h.

Unfortunately the vertical system happened not to be in operation on March 7. Graph 11 shows the results in 6-hour mean values. It is obvious that the unshielded vessel received more surplus radiation (6 %) than the one shielded with 110 cm Fe (4 %). But this unshielded instrument 6 registers only two-hour values, which entails that the maximum is less pronounced than when computed from one-hour values, like those given for both the vessels 5 and 8, the curves of which may be compared with each other. It is in accordance with this statement that an increase of 9 % is found in the vessel 5 under a shield of 12 cm Fe. From the difference between the vessels 6 and 8 we may infer that on this occasion more than one third of the total amount of particles with an energy $> 4 \cdot 10^9$ eV (the energy required to penetrate the atmosphere) were of an energy $< 5 \cdot 10^9$ eV (the energy required to penetrate the atmosphere and an additional shield of 110 cm Fe).

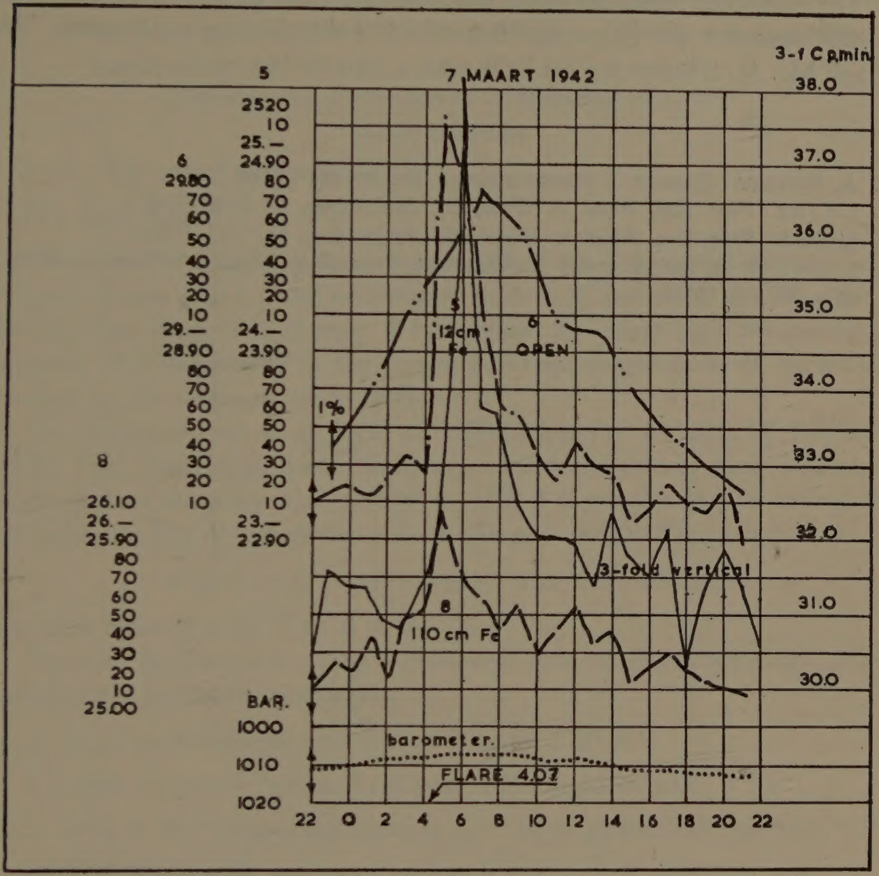
Moreover in the same graph the 6 hour mean values are given for 3 fold coincidences in a vertical direction with the unshielded counter system. The increase is from 30,8 to 38,2 p. minute i.e. about 24 %.



Graph. 11. Intensity curves of cosmic radiation (6 hour means) during the period 1—10 March 1942. At left scales of the curves. Time of observation of solar flare is indicated, 4 h. 07.

From 2—6 March the counter system which measured 2 fold coincidences in a horizontal direction had a considerable decrease presumably on account of the magnetic storm of 1—2 March, the effect of which is felt also in the ionisation vessels in that period. On the 7th however, during the increase in the vessels and the vertical set correlated with the solar flare, no increase manifests itself in the horizontal direction. To be registered here the minimum energy of the excess radiation would have to be of the order of 2×10^{10} eV, on account of the absorption of the atmosphere. No particles of this energy seem to have been produced by the sun on this occasion.

In graph 12 we see the curve for 7 March more in detail, the hour-values being given (for 6 the 2 hour-values). We note that for the most energetic particles (8) the maximum is reached earlier than for the softer particles (6) and the counters. As our records only contain hour-values, we dare not yet draw quantitative conclusions regarding this difference in time.



Graph. 12. Intensity curves of cosmic radiation (hour values, for 6 the 2 hour values) on March 7, 1942.

Compared with our results, EHMERT's total number of effects strikes us as being rather small for a period of several years and the same applies to the number of cases which he signalizes as having a possible correlation with the sun.

To conclude we wish to point out the curious circumstance, that many of these sudden increases happen in the months of May, June and July vide part I and II. An instance of the variation under discussion had already been observed by us in May 1927 in two independent experimental arrangements. We reported about it at the time [2]). In May 1928 we registered again a temporary increase under a shield of 50 cm Pb [3]). In June 1936 two of our ionisationvessels as well as a counter system recorded an abrupt increase of 10 % [4]). The effects observed by EHMERT in the period 1941—1943 occurred roughly taken in the same time of the year.

The conviction obtrudes itself, that this cannot be entirely accidental, but further research will be needed to arrive at a satisfactory explanation. We thank Mr. H. SWIERS for his help with a part of the observations.

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Mathematics. — *The second pearl of the theory of numbers.* By J. G. VAN DER CORPUT and J. H. B. KEMPERMAN. (Third communication.)

(Communicated at the meeting of September 24, 1949.)

§ 3. *The proof of theorem 7.*

We state again:

Theorem 7. *Let be defined on an ordered set G a partial addition with respect to two given finite non-empty subsets A and B of G. We denote the smallest element of A resp. B by* a_0 *resp.* b_0*, where* b_0 *is the particular element of B mentioned in the fourth addition property; let B′ be the set formed by the elements* $\neq b_0$ *of B.*

Let be defined on G a finite or infinite number of pairs of functions $\varphi(g)$, $f(g)$*, such that, for an arbitrary considered pair of functions* $\varphi(g)$, $f(g)$*, and an arbitrary element* $b \neq b_0$ *of B it is possible to find among the considered pairs of functions at least one pair* $\Phi(g)$, $F(g)$*, such that*

$$\varphi(g+b) \leq \Phi(g) + \varphi(b); \quad f(g+b) \geqq F(g)$$

for each element g of A or $A+B$.

Suppose further that every considered pair of functions $\varphi(g)$, $f(g)$*, satisfies the following conditions:*

1. $\varphi(g)$ *is monotonic non-decreasing and* $f(g) \geqq 0$.
2. $\varphi(g) = \varphi(g+b_0)$; $\varphi(b) \leq \varphi(g+b)$; $\varphi(b) = \varphi(a_0+b)$.
3. *If* $A+B'$ *contains the element* a_0+b_0 *or at least one element* $< a_0$*, then* $\varphi(a_0) \leq 0$.
4. $f(g) = f(g+b_0)$; $f(b) \leq f(g+b)$.

Here g denotes an arbitrary element of A or $A+B$ *and b an arbitrary element* $\neq b_0$ *of B.*

If under these conditions each pair of functions $\varphi(g)$, $f(g)$ *satisfies the inequality*

$$A(h) + B'(h) \geqq \varphi(h)$$

for each element h of a given subset H of G, which contains all elements $> a_0$ *of A and all elements* $> b_0$ *of B, then we have for each considered pair of functions* $\varphi(g)$, $f(g)$*, and for each element* $h > a_0 + b_0$ *of H*

$$(A+B)(h) \geqq \varphi(h).$$

Here $A(h)$, $B'(h)$ *and* $(A+B)(h)$ *are defined by* (14).

Proof. We assume for the present that B' contains at least one element. Let b be such an element. By the fourth addition property A contains at least one element a such that no element x of A exists with $a+b = x+b_0$. In fact, if this was not true, to any element a of A would correspond an element a' of A with $a + b = a' + b_0$. Starting from an arbitrary element y_0 of A we find successively

$$y_\mu + b = y_{\mu+1} + b_0 \qquad (\mu = 0, 1, 2, \dots),$$

where $y_0, y_1, \dots$ are elements of A. Since A is a finite set, at least one element occurs twice in this sequence, hence $y_p = y_{p+m}$, where $p \geq 0$ and $m > 0$ are suitable integers. The elements

$$g_\mu = y_{p+\mu} + b \qquad (\mu = 0, 1, \dots m)$$

belong to $A + B$, and satisfy for $\mu = 0, 1, \dots, m-1$

$$g_\mu + b = (y_{p+\mu} + b) + b = (y_{p+\mu+1} + b_0) + b = (y_{p+\mu+1} + b) + b_0 = g_{\mu+1} + b_0$$

which contradicts the fourth addition property by $g_m = g_0$.

Consequently A contains at least one element $\bar{a}$ with the property that it is possible to find in B' at least one element b, such that A does not contain an element x with $\bar{a} + b = x + b_0$. Let $\bar{a}$ be the smallest element of A with this property; this element is called by us the *basic element* of the pair A, B'.

Lemma 3. *Under the conditions of theorem 7 we have*

$$A(a) \geq \varphi(a)$$

for each element a of A with $a_0 < a \leq \bar{a}$ and for each considered pair of functions $\varphi(g)$ and $f(g)$.

Proof. Suppose $\bar{a} > a_0$, for otherwise there is nothing to be proved. The smallest element b' of B' satisfies the relation $B'(b') = 0$ and therefore

$$A(b') \geq \varphi(b').$$

The assertion of lemma 3 is obvious for the elements a of A with $a_0 < a \leq b'$, because then $B'(a) = 0$ and $A(a) = A(a) + B'(a) \geq \varphi(a)$.

In the proof we may therefore suppose not only $a_0 < a \leq \bar{a}$, but also $a > b'$.

We will deduce the inequality to be proved for $a = a_1$, where a_1 denotes an element of A with $a_0 < a_1 \leq \bar{a}$ and $a_1 > b'$, under the assumption that the inequality has already been proved for each considered pair of functions $\varphi(g)$, $f(g)$, and for all elements a of A lying between a_0 and a_1.

We say that an element a of A possesses a corresponding element a' if a' belongs to A and satisfies $a + b' = a' + b_0$. From the second addition property it follows that the element a', if it exists, is uniquely defined by a. By the minimum property of $\bar{a}$ each element $a < \bar{a}$ of A possesses a corresponding element.

We shall show that A contains at least one element $< a_1$ with corresponding element $\geq a_1$. The element a_0 is less than $a_1 \leq \bar{a}$ and possesses therefore a corresponding element z_1. If $z_1 \geq a_1$ we are ready. If not, z_1 possesses a corresponding element z_2. If $z_2 \geq a_1$ we are ready again, and if not, z_2 possesses a corresponding element z_3 etc. Continuing in this manner we find a sequence of elements $a_0, z_1, z_2, \dots$ of A. In the same manner as above (with $a_0, z_1, \dots$ in stead of $y_0, y_1, \dots$) we obtain that this sequence can not be continued infinitely, so that it contains a last element which is $\geqq a_1$. Therefore $z_m < a_1 \leqq z_{m+1}$, if m is suitably chosen, so that z_m possesses the required properties.

Let a_2 be the smallest element $< a_1$ of A with the property that the corresponding element a'_2 is $\geqq a_1$, hence

$$a_2 < a_1 \leqq a_2' \quad \text{and} \quad a_2 + b' = a_2' + b_0 . \quad . \quad . \quad . \quad . \quad (22)$$

Each element $a < a_2$ of A possesses by $a_2 < a_1 \leqq \bar{a}$ a corresponding element a'; this element is $< a_1$, for otherwise the relations (22) would remain valid, if we replace a_2 by a and a_2' by a', which contradicts the minimum property of a_2.

We distinguish now two cases.

1. Let us consider first the case $a_2 > a_0$. In this case A contains elements $< a_2$. Let a_3 be the element $< a_2$ of A for which the corresponding element a_3' is minimum. By $a_3 < a_2$ this element a_3' of A is $< a_1$. Further it follows from the definition of a_3 that for each element $a < a_2$ of A the corresponding element a' is $\geqq a_3'$, hence $a_3' \leqq a' < a_1$. From the second addition property we conclude not only that the corresponding element a' is defined uniquely by a, but also conversely that different elements a furnish different corresponding elements. Consequently the sum $\sum\limits_{a<a_2} f(a')$, where a' denotes the corresponding element of A, is equal to the sum $\Sigma' f(a')$, extended over the elements a' of A with $a_3' \leqq a' < a_1$ which correspond to an element $< a_2$ of A. Since the function $f(g)$ is $\geqq 0$, we obtain

$$\sum_{a<a_2} f(a') \leqq \sum_{a_3' \leqq a < a_1} f(a) = A(a_1) - A(a_3') . \quad . \quad . \quad . \quad . \quad (23)$$

By the conditions of theorem 7 we find

$$f(a') = f(a' + b_0) = f(a + b') \geqq F(a),$$

where the pair of functions $\Phi(g)$, $F(g)$, depending on b', is chosen as is indicated in the enunciation of theorem 7. Consequently

$$\Phi(a_2) + \varphi(b') \geqq \varphi(a_2 + b') . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

In virtue of $a_0 < a_2 < a_1 \leqq \bar{a}_1$ we find by induction

$$\sum_{a<a_2} f(a') \geqq \sum_{a<a_2} F(a) \geqq \Phi(a_2) . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Moreover we shall prove

$$A(a_3') \geqq \varphi(b') . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The inequality

$$A(a_3') + B(a_3') \geqq \varphi(a_3')$$

is obvious for the case $a_3' > a_0$, as a_3' is an element of A. If on the contrary

$a_3' = a_0$, then $A + B'$ contains the element $a_0 + b_0 = a_3' + b_0 = a_3 + b'$, hence $\varphi(a_0) \leqq 0$ from the conditions of theorem 7.

In virtue of

$$\varphi(a_3') = \varphi(a_3' + b_0) = \varphi(a_3 + b') \geqq \varphi(b')$$

we find for the case $a_3' \leq b'$

$$A(a_3') = A(a_3') + B(a_3') \geqq \varphi(a_3') \geqq \varphi(b').$$

If on the contrary $a_3' > b'$ we have

$$A(a_3') \geqq A(b') = A(b') + B'(b') \geqq \varphi(b'),$$

so that (26) is obvious. Hence by (23), (25) and (26)

$$\begin{aligned} A(a_1) &\geqq \Phi(a_2) + \varphi(b') \geqq \varphi(a_2 + b') && \text{by (24)} \\ &= \varphi(a_2' + b_0) = \varphi(a_2') \geqq \varphi(a_1) && \text{by (22)}, \end{aligned}$$

since $\varphi(g)$ is a monotonic function. This establishes lemma 3 in this case.

2. Finally we consider the case $a_2 = a_0$. From (22) we deduce $a_1 \leq a_0'$, where $a_0' = a_2'$ is the corresponding element of a_0. Then we have

$$\varphi(a_1) \leq \varphi(a_0') = \varphi(a_0' + b_0) = \varphi(a_0 + b') = \varphi(b').$$

In virtue of $a_1 > b'$ we find therefore

$$A(a_1) \geqq A(b') = A(b') + B'(b') \geqq \varphi(b') \geqq \varphi(a_1).$$

This completes the proof of lemma 3.

Let us denote by $b_1, \ldots, b_k$ the elements b of B', such that A does not contain an element x with $\bar{a} + b = x + b_0$. From the definition of the basic element $\bar{a}$ it follows that $k \geqq 1$.

We introduce k new elements $x_1, \ldots, x_k$ and we say that an element g of G is $< x_l$ or $> x_l$ according to whether $g \leqq \bar{a} + b_l$ or $> \bar{a} + b_l$ $(l = 1, 2, \ldots, k)$. Further we agree to write $x_l > x_m$ if $\bar{a} + b_l > \bar{a} + b_m$ (in the last inequality the sign of equality is excluded by the third addition property). Let G_k be the set formed by the elements of G and the elements $x_1, \ldots, x_k$. By considering the different possible cases we convince ourselves of the fact that G_k is ordered.

Each of the considered pairs of functions $\varphi(g)$ and $f(g)$ is defined not only for every element g of G, but also for the elements $x_1, \ldots, x_k$ of G_k, which do not belong to G, by the convention

$$\varphi(x_l) = \varphi(\bar{a} + b_l) \quad ; \quad f(x_l) = f(\bar{a} + b_l). \quad . \quad . \quad . \quad . \quad (27)$$

Consequently $\varphi(g^*)$ and $f(g^*)$ are defined on G_k, such that $\varphi(g^*)$ is monotonic non-decreasing and $f(g^*)$ is $\geqq 0$.

By canceling b_1 we transform B into a set B_1; by canceling b_2 we transform B_1 into a set B_2, ..., by canceling finally b_k we transform B_{k-1} into a set B_k. By adding x_1 we transform A and H into the sets A_1 and H_1; by adding x_2 we transform A_1 and H_1 into the sets A_2 and H_2; ...; by adding finally x_k we transform A_{k-1} and H_{k-1} into the sets A_k and H_k.

It is obvious that H_k contains every element $\neq a_0$ of A_k and every element $\neq b_0$ of B_k. By B'_l we denote the set formed by the elements $\neq b_0$ of B_l $(l = 1, 2, \ldots, k)$.

Lemma 4. *Under the conditions of theorem 7 we have*

$$A_l(h) + B'_l(h) \geqq \varphi(h)$$

for each element h of $H_l (l = 1, 2, \ldots, k)$ and for each considered pair of functions $\varphi(g)$ and $f(g)$.

Proof. If $l \geqq 2$ we may assume

$$A_{l-1}(h) + B'_{l-1}(h) \geqq \varphi(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

for each element h of H_{l-1}, and each considered pair of functions. This inequality holds also for $l = 1$, if A_0, B_0' and H_0 denote the sets A, B' and H.

We distinguish three cases.

1. Let h be an element $> x_l$ of H_l. Then

$$A_{l-1}(h) + f(x_l) = A_l(h) \quad \text{and} \quad B'_{l-1}(h) \leqq B'_l(h) + f(b_l),$$

where

$$f(x_l) = f(\bar{a} + b_l) \geqq f(b_l).$$

The element h of H_l is greater than the element x_l, which has been added to H_{l-1}, so that h belongs to H_{l-1}, and (28) may be applied, which gives the assertion of lemma 2.

2. Let h be an element $\leqq b_l$ of H_l. For this element we shall deduce first inequality (28). For that purpose we may restrict ourselves to the case $h = x_l$, so that is sufficient to prove

$$A_{l-1}(x_l) + B_{l-1}(x_l) \geqq \varphi(x_l), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

under the assumption $x_l \leqq b_l$.

If $x_l < a_0$ we have $\bar{a} + b_l < a_0$ by the definition of the order of the element x_l with respect to the elements of G. Consequently $A + B'$ contains at least one element $< a_0$, hence $\varphi(a_0) \leqq 0$ by the conditions of theorem 7. Therefore the monotonic non-decreasing function $\varphi(g^*)$ satisfies the inequality $\varphi(x_l) \leqq 0$ and (29) is evident.

If $x_l > a_0$ we consider the smallest element $h^* \geqq x_l$, which belongs to A_{l-1} or B'_{l-1}. This element exists by $b_l \geqq x_l$. It may be observed that h^* belongs to H_{l-1}. This is clear if h^* belongs to B'_{l-1}, as B'_{l-1} is a subset of B', therefore of H, consequently of H_{l-1}. Moreover if h^* were an element of A_{l-1} not belonging to H_{l-1}, then h^* does not coincide with one of the elements $x_1, \ldots, x_{l-1}$, which have been added in the construction of A_{l-1} and H_{l-1}. In that case h^* would belong to A and not to H, but a_0 is the only element which may have this property and this element does not enter into consideration by $h^* \geqq x_l > a_0$.

Since h^* is an element of H_{l-1} we have

$$A_{l-1}(x_l)+B'_{l-1}(x_l)=A_{l-1}(h^*)+B'_{l-1}(h^*)\geqq\varphi(h^*)\geqq\varphi(x_l).$$

The case $x_l=a_0$ is excluded, as x_l is a new element not belonging to G. This establishes (29) and therefore also (28) for each element $h\leqq b_l$ of H_l and each considered pair of functions. The inequalities (28) furnish the assertion of lemma 4 by

$$A_{l-1}(h)\leqslant A_l(h) \quad\text{and}\quad B'_{l-1}(h)=B'_l(h).$$

3. Let h be an element of H_l with $b_l<h\leqq x_l$, and let $\varphi(h)$, $f(h)$ denote a fixed considered pair of functions. The inequality

$$\varphi(h)\leqq\varphi(a_4+b_l) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

is by $h\leqq x_l$ and $\varphi(x_l)=\varphi(\bar{a}+b_l)$ valid, if we replace a_4 by $\bar{a}$. The smallest element a_4 of A with (30) is therefore $\leqq\bar{a}$. From $a_4\leqq\bar{a}$ and the minimum property of $\bar{a}$ it follows for each element $a<a_4$ of A that A contains an element a' with $a+b_l=a'+b_0$. This element a' is $<h$, for if not, we should have

$$\varphi(a+b_l)=\varphi(a'+b_0)=\varphi(a')\geqq\varphi(h)$$

contrary to the minimum property of a_4.

If $a_4=a_0$, we obtain by (30)

$$\varphi(h)\leqq\varphi(a_0+b_l)=\varphi(b_l),$$

hence by means of $h>b_l$, where b_l is an element of H_{l-1},

$$A_l(h)+B'_l(h)\geqq A_l(b_l)+B'_l(b_l)\geqq A_{l-1}(b_l)+B'_{l-1}(b_l)\geqq\varphi(b_l)\geqq\varphi(h),$$

which gives in this case the assertion to be proved.

Let us finally consider the case $a_4>a_0$. Then A contains at least one element $<a_4$. Let a_5 be the element $<a_4$ of A, such that the element a_5' of A, defined by $a_5+b_l=a_5'+b_0$, is as small as possible. By the second addition property different elements $a<a_4$ of A furnish different elements a' of A with $a+b_l=a'+b_0$ and $a_5'\leqq a'<h$. Consequently the sum $\sum\limits_{a<a_4} f(a')$, where a' denotes the element defined by $a+b_l=a'+b_0$, is equal to the sum $\Sigma' f(a')$ extended over the elements a' of A with $a_5'\leqq a<h$, which correspond to an element $a<a_4$, of A by means of the relation $a+b_l=a'+b_0$. As the function $f(g)$ is $\geqq 0$, we obtain therefore

$$\sum_{a<a_4} f(a')\leqq\sum_{a_5'\leqq a<h} f(a)=A(h)-A(a_5')\leqq A_l(h)-A_l(a_5'),$$

because A is a subset of A_l. It is possible to find among the considered pairs of functions a pair $\Phi(g)$, $F(g)$, such that

$$\Phi(a_4)+\varphi(b_l)\geqq\varphi(a_4+b_l) \quad . \; . \; . \; . \; . \; . \quad (31)$$

and further

$$\sum_{a<a_4} f(a')=\sum_{a<a_4} f(a'+b_0)=\sum_{a<a_4} f(a+b_l)\geqq\sum_{a<a_4} F(a).$$

The right hand side of the last formula is $\geqq \Phi(a_4)$ by lemma 3, which may be applied here on account of $a_0 < a_4 \leqq \bar{a}$. In this manner we obtain

$$A_l(h) \geqq A_l(a_5') + \Phi(a_4),$$

and therefore by $h > b_l$

$$A_l(h) + B_l'(h) \geqq A_l(a_5') + B_l'(h) + \Phi(a_4) \geqq A_{l-1}(a_5') + B_{l-1}'(b_l) + \Phi(a_4).$$

Using the inequalities

$$\Phi(a_4) + \varphi(b_l) \geqq \varphi(a_4 + b_l) \geqq \varphi(h),$$

which follow from (31) and (30), we are ready if we have proved

$$A_{l-1}(a_5') + B_{l-1}'(b_l) \geqq \varphi(b_l). \qquad \qquad (32)$$

First we remark that $\varphi(a_5') \geqq \varphi(b_l)$ by

$$\varphi(a_5') = \varphi(a_5' + b_0) = \varphi(a_5 + b_l) \geqq \varphi(b_l).$$

If $a_5' = a_0$ the element $a_0 + b_0 = a_5 + b_l$ belongs to $A + B'$, hence $\varphi(a_0) \leqq 0$ from the conditions of theorem 7, consequently $\varphi(b_l) \leqq \varphi(a_0) \leqq 0$, so that in this case (32) is obvious. Therefore we may suppose $a_5' > a_0$, so that the element a_5' of A belongs to H, and therefore as well to H_{l-1}. The set H_{l-1} contains also b_l, and therefore h_1, if h_1 is the smallest of the two elements a_5' and b_l. By $\varphi(a_5') \geqq \varphi(b_l)$ we have $\varphi(h_1) \geqq \varphi(b_l)$. Consequently

$$A_{l-1}(a_5') + B_{l-1}'(b_l) \geqq A_{l-1}(h_1) + B_{l-1}'(h_1) \geqq \varphi(h_1) \geqq \varphi(b_l),$$

which establishes the proof of lemma 4.

Definition of a partial addition with respect to the sets A_k, B_k. For each of the elements $x_1, \ldots, x_k$ and for each element b of B_k we define a sum $x_k + b$ in the following manner: as b does not coincide with any of the elements $b_1, \ldots, b_k$, the set A contains an uniquely defined element a' with $\bar{a} + b = a' + b_0$, and we put by definition $x_l + b = a' + b_l$.

From this definition it appears that $x_l + b$ belongs to $A + B$. In particular we find

$$x_l + b_0 = \bar{a} + b_l. \qquad \qquad (33)$$

We shall prove that we have defined in this manner on G_k a partial addition with respect to the subsets A_k and B_k, such that b_0 is again the special element occurring in the fourth addition property. This proof runs as follows.

For each element a^* of A_k and each element b of B_k we have an uniquely defined sum $a^* + b$. In fact, if a^* belongs to A we use the old definition and otherwise a^* coincides with one of the elements $x_1, \ldots, x_k$, so that we make use of the new definition; the sum $a^* + b$ found in this manner belongs to $A + B$, therefore to G and also to G_k. Moreover we obtain for each element g^* of $A_k' + B_k$ and each element b of B_k an uniquely defined element $g^* + b$ belonging to G and therefore to G_k, because g^* belongs to $A + B$. Further the four following addition properties are valid.

1. If a^* is an element of A_k and b and b' are elements of B_k, then

$$(a^* + b) + b' = (a^* + b') + b.$$

This relation is obvious if a^* belongs to A, as b and b' belong to B_k and therefore also to B. Therefore we may suppose $a^* = x_l$, where $1 \leq l \leq k$. By definition we have $x_l + b = a + b_l$, where a denotes the element of A with $\bar{a} + b = a + b_0$. Similarly $x_l + b' = a' + b_l$, where $\bar{a} + b' = a' + b_0$. From these relations it follows by the first addition property

$$(a+b')+b_0 = (a+b_0)+b' = (\bar{a}+b)+b' = \\ = (\bar{a}+b')+b = (a'+b_0)+b = (a'+b)+b_0,$$

hence by the second addition property $a + b' = a' + b$. Consequently

$$(x_l+b)+b' = (a+b_l)+b' = (a+b')+b_l = \\ = (a'+b)+b_l = (a'+b_l)+b = (x_l+b')+b.$$

2. If b is an element of B_k and g^* and g^{**} belong either both to A_k or both to $A_k + B_k$, then

$$g^* + b = g^{**} + b \text{ implies } g^* = g^{**}.$$

This relation is obvious if both g^* and g^{**} belong to A. Also if they belong both to $A_k + B_k$ and therefore to $A + B$. Consequently we may suppose that one of the two elements g^* and g^{**} coincides with one of the elements x_l $(1 \leq l \leq k)$, and that the other of the two elements belongs to A_k. Without loss of generality we may assume $g^{**} = x_l$ and that g^* belongs to A_k. First we shall prove that g^* does not belong to A. By definition we have $x_l + b = a + b_l$, where a denotes the uniquely defined element of A with $\bar{a} + b = a + b_0$. If g^* were an element of A we should have

$$(g^*+b_0)+b = (g^*+b)+b_0 = (x_l+b)+b_0 \\ = (a+b_l)+b_0 = (a+b_0)+b_l = (\bar{a}+b)+b_l = (\bar{a}+b_l)+b$$

and therefore by the second addition property $g^* + b_0 = \bar{a} + b_l$, which is excluded, as it follows from the definition of b_l that A does not contain an element x with $x + b_0 = \bar{a} + b_l$. Consequently g^* does not belong to A and coincides therefore with one of the elements x_m $(1 \leq m \leq k)$. Then $x_m + b = a + b_m$, where a is the above mentioned element of A with $\bar{a} + b = a + b_0$. From $x_l + b = x_m + b$ we deduce $a + b_l = a + b_m$, therefore by the third addition property $b_l = b_m$, hence $l = m$, consequently $x_l = x_m$.

3. If a^* is an element of A_k, and b and b' are elements of B, then

$$a^* + b = a^* + b' \text{ implies } b = b'.$$

We may suppose that a^* does not belong to A, since otherwise the assertion follows from the third addition property. Therefore we may assume $a^* = x_l$ $(1 \leq l \leq k)$. Then

$$x_l + b = a + b_l \quad \text{and} \quad x_l + b' = a' + b_l, \text{ where}$$
$$\bar{a} + b = a + b_0 \quad \text{and} \quad \bar{a} + b' = a' + b_0.$$

From $x_l + b = x_l + b'$ follows $a + b_l = a' + b_l$, hence by the second addition property $a = a'$, therefore $\bar{a} + b = \bar{a} + b'$, consequently $b = b'$ by the third addition property.

4. The element b_0 of B mentioned in the fourth addition property satisfies the following condition: it is impossible to find in B_k an element $b \neq b_0$, and in $A_k + B_k$ a finite number of elements

$$g_0^*,\ g_1^*, \ldots, g_{m-1}^*,\ g_m^* = g_0^* \quad (\text{where } m \geqq 1)$$

with the properties

$$g_\mu^* + b = g_{\mu+1}^* + b_0 \quad (\mu = 0, 1, \ldots, m-1).$$

This assertion follows immediately from the fourth addition property, as $A_k + B_k$ is a subset of $A + B$.

Properties of the functions $\varphi(g^*)$ and $f(g^*)$.

We shall show that all conditions of theorem 7 remain valid if we replace G, A and B resp. by G_k, A_k and B_k. We know that G_k is ordered and, as we have proved above, we have defined on G_k a partial addition with respect to the subsets A_k and B_k of G_k. The smallest element b_0 of B and therefore of B_k is just the special element of B_k mentioned in the fourth addition property. Finally we must prove that the considered pairs of functions $\varphi(g^*)$ and $f(g^*)$ possess on G_k the required properties. We have already observed that $\varphi(g^*)$ is monotonic non-decreasing on G_k and that $f(g^*) \geq 0$. We shall now deduce the relations:

$$\varphi(g^*) = \varphi(g^* + b_0) \quad \text{and} \quad f(g^*) = f(g^* + b_0)\,; \quad \ldots\ldots \quad (34)$$

$$\varphi(b) \leqq \varphi(g^* + b) \quad \text{and} \quad \varphi(g^* + b) \leqq \Phi(g^*) + \varphi(b)\,; \quad \ldots \quad (35)$$

$$f(b) \leqq f(g^* + b) \quad \text{and} \quad f(g^* + b) \geqq F(g^*)\,; \quad \ldots\ldots \quad (36)$$

here g^* is an arbitrary element of A_k or $A_k + B_k$, further b is an arbitrary element $\neq b_0$ of B_k, and finally $\Phi(g^*)$ and $F(g^*)$ denote the pair of functions, which is mentioned in theorem 7 and which depends on b.

As $A_k + B_k$ is a subset of $A + B$, and B_k is a subset of B, it is sufficient to consider the case that g^* is equal to an element x_l of A_k, which does not belong to A. By (27) and (33) we have

$$\varphi(x_l) = \varphi(\bar{a} + b_l) = \varphi(x_l + b_0),$$
$$f(x_l) = f(\bar{a} + b_l) = f(x_l + b_0),$$

which give the relations (34).

We have $x_l + b = a + b_l$, where a denotes the element of A with $\bar{a} + b = a + b_0$, hence

$$(a + b_l) + b_0 = (a + b_0) + b_l = (\bar{a} + b) + b_l = (\bar{a} + b_l) + b,$$

therefore

$$\varphi(x_l + b) = \varphi((x_l + b) + b_0) = \varphi((\bar{a} + b_l) + b),$$

consequently

$$\varphi(b) \leq \varphi(x_l + b) \leq \Phi(\bar{a} + b_l) + \varphi(b) = \Phi(x_l) + \varphi(b)$$

by the definition of $\Phi(x_l)$. In this manner we find (35) and similarly (36).

The reader should observe that each element $a^* + b$ of $A_k + B'_k$ belongs to $A + B'$. This is obvious if a^* belongs to A, as B'_k is a subset of B'; otherwise we have $a^* = x_l$ and the element $x_l + b = a + b_l$ belongs to $A + B'$.

Finally we prove that

$$\varphi(b) = \varphi(a_0^* + b). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

(where a_0^* denotes the smallest element of A_k, and b an arbitrary element $\neq b_0$ of B_k), and moreover $\varphi(a_0^*) \leqq 0$, if $A_k + B'_k$ contains the element $a_0^* + b_0$ or at least one element $< a_0^*$.

This assertion is obvious in the special case $a_0^* = a_0$, as $A_k + B'_k$ is a subset of $A + B'$. In the remaining case a_0^* is less than a_0 and is therefore equal to one of the elements x_l belonging to A_k but not to A. By the definition of the order of G_k the inequality $x_l < a_0$ implies $\bar{a} + b_l < a_0$, so that $A + B'$ contains at least one element $< a_0$, hence $\varphi(a_0) \leqq 0$ by the first condition of theorem 5, consequently the monotonic non-decreasing function $\varphi(g^*)$ satisfies the inequality $\varphi(a_0^*) \leqq 0$. This result is valid for each considered pair of functions $\varphi(g^*)$ and $f(g^*)$. By (35) we have for each element $b \neq b_0$ of B_k

$$\varphi(b) \leqq \varphi(a_0^* + b) \leqq \Phi(a_0^*) + \varphi(b).$$

The above result applied with the pair $\Phi(g^*)$, $F(g^*)$, in stead of $\varphi(g^*)$, $f(g^*)$, gives $\Phi(a_0^*) \leqq 0$, hence $\varphi(b) \leqq \varphi(a_0^* + b) \leqq \varphi(b)$, so that we find (37).

The reader should observe that the above argument has shown $\varphi(a_0^*) \leqq 0$, if $a_0^* \neq a_0$; in that case we obtain for each element $h \leqq a_0^* + b_0$ of G_k

$$\varphi(h) \leqq \varphi(a_0^* + b_0) = \varphi(a_0^*) \leqq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

End of the proof of theorem 7.

Let us first consider the case that B contains only the element b_0. Then B' is empty and $B'(h)$ is always equal to 0. If the inequalities

$$A(h) + B'(h) \geqq \varphi(h)$$

are valid for each considered pair of functions $\varphi(g)$, $f(g)$, and for each element h of a subset H of G, which contains all elements $> a_0$ of A, then we have for these h

$$A(h) \geqq \varphi(h). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If the assertion of theorem 7 was not true, H would contain an element $h > a_0 + b_0$ with

$$(A + B)(h) < \varphi(h) \leqq A(h),$$

where the pair of functions $\varphi(g)$ and $f(g)$ is suitably chosen. Consequently

$$\sum_{a + b_0 < h} f(a + b_0) < \sum_{a < h} f(a).$$

An element a, which furnishes to both sides a contribution, has by $f(a+b_0) = f(a)$ the property that these contributions are equal. An element a, which does not furnish a contribution to the righthand side, has the property that its contribution to the lefthand side is $\geqq 0$. Consequently A would contain at least one element a_6, which gives a contribution to the righthand side, but not to the lefthand side. This element a_6 would satisfy the inequalities

$$a_6 < h \leqq a_6 + b_0. \quad \ldots \ldots \ldots \quad (40)$$

Let a_6 be the smallest element of A with these properties. In virtue of $h > a_0 + b_0$ we have $a_6 \neq a_0$ and therefore $a_6 > a_0$. Each element $a < a_6$ of A satisfies $a + b_0 < h$ by the minimumproperty of a_6, and different elements $a < a_6$ of A furnish different elements $a + b_0 < h$. Consequently by $f(g) \geqq 0$

$$(A+B)(h) = \sum_{a+b_0<h} f(a+b) \geqq \sum_{a<a_6} f(a+b_0) =$$
$$= \sum_{a<a_6} f(a) \geqq \varphi(a_6) = \varphi(a_6+b_0) \geqq \varphi(h)$$

by (40) and the monotony of $\varphi(g)$.

Finally we assume that B' contains at least one element. We may suppose that the theorem has already been proved if the number of elements of B' is replaced by a smaller number $\geqq 0$. Since B' contains at least one element we may apply lemma 4, hence

$$A_k(h) + B_k'(h) \geqq \varphi(h) \quad \ldots \ldots \ldots \quad (41)$$

for each element h of H_k and for each considered pair of functions $\varphi(g)$ and $f(g)$.

In the special case that the smallest element a_0^* of A is $< a_0$ we have $a_0^* = x_l$ $(1 \leq l \leq k)$, so that $A + B'$ contains the element $\bar{a} + b_0 < a_0$ and from the conditions of theorem 7 we find $\varphi(a_0) \leqq 0$, so that (41) is valid for $h = a_0$. In this case we transform H_k into a set H'_k by adding the element a_0. If $a_0^* = a_0$ we put $H'_k = H_k$. Now we have that the inequality (41) is valid for each considered pair of functions $\varphi(g)$ and $f(g)$ and for each element h of the subset H'_k of G_k, which contains all elements $> a_0^*$ of A_k and all elements $> b_0$ of B_k.

Moreover we know that all conditions of theorem 7 are satisfied if we replace G, A and B resp. by G_k, A_k and B_k. Consequently we find by induction

$$(A_k + B_k)(h) \geqq \varphi(h)$$

for each considered pair of functions $\varphi(g)$ and $f(g)$ and for each element $h > a_0^* + b_0$ of H'_k, where a_0^* is the smallest element of A_k. Since $A_k + B_k$ is a subset of $A + B$, and further H is a subset of H'_k, we obtain the assertion of theorem 7 for each element $h > a_0^* + b_0$ of H. We are ready if $a_0^* = a_0$, otherwise the assertion follows from (38).

Mathematics. — *On the geometry of spin spaces.* III. By J. A. SCHOUTEN.

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§ 5. *Invariants and transformations for* $\nu = 1$.

For $\nu = 1$ we write r, c, k, ω, π instead of R, C, K, Ω, Π. From (3.10) it is easily derived that, the first invariant condition being satisfied

$$c :: ({}^1{}_2 + {}^2{}_1) \quad ; \quad k = cr :: ({}^1{}_2 - {}^2{}_1) \qquad \ldots \ldots \quad (5.1)$$

independently of the choice of λ. For ω we get, according to the condition that it is hermitian with a determinant ± 1:

$$\left.\begin{array}{lll}
\underset{1}{\varepsilon} = 1, \quad \underset{2}{\varepsilon} = 1 & : & \omega(1,1) = \pm\, ({}^1/_{\lambda_m}\, {}^{\bar{1}1} + \lambda_m {}^{\bar{2}2}); \\
\underset{1}{\varepsilon} = 1, \quad \underset{2}{\varepsilon} = i & : & \omega(1,i) = \pm\, (\lambda_a\, {}^{\bar{1}2} + {}^1/_{\lambda_a}\, {}^{\bar{2}1}); \\
\underset{1}{\varepsilon} = i, \quad \underset{2}{\varepsilon} = 1 & : & \omega(i,1) = \pm\, i\, (\lambda_a\, {}^{\bar{1}2} - {}^1/_{\lambda_a}\, {}^{\bar{2}1}); \\
\underset{1}{\varepsilon} = i, \quad \underset{2}{\varepsilon} = i & : & \omega(i,i) = \pm\, ({}^1/_{\lambda_m}\, {}^{\bar{1}1} - \lambda_m {}^{\bar{2}2});
\end{array}\right\} \quad (5.2)$$

$$\lambda_m \overset{\text{def}}{=} |\lambda| \quad ; \quad \lambda_a \overset{\text{def}}{=} \frac{\lambda}{|\lambda|} \; ; \; \left(\lambda_a^2 = \frac{\lambda}{\bar{\lambda}}\right) \quad ; \quad \lambda = \lambda_m \lambda_a .$$

Now be

$$\alpha^1{}_1 + \beta^1{}_2 + \gamma^2{}_1 + \delta^2{}_2 \qquad \ldots \ldots \ldots \quad (5.3)$$

a transformation in $\mathfrak{S}_2$ leaving c invariant and r invariant or invariant to within the sign. Then it is easily proved that there are only two possibilities:

$$\left.\begin{array}{ll}
\text{I} & \alpha^1{}_1 + 1/\alpha\, {}^2{}_2 \; ; \; \beta = 0 \; ; \; \gamma = 0 \; ; \; \underset{s}{\Delta} = +1 \; ; \; c \to c \; ; \; r \to r \\
\text{II} & \beta^1{}_2 + 1/\beta\, {}^2{}_1 \; ; \; \alpha = 0 \; ; \; \delta = 0 \; ; \; \underset{s}{\Delta} = -1 \; ; \; c \to c \; ; \; r \to -r
\end{array}\right\} \quad (5.4)$$

with arbitrary *real or complex* parameters α and β. Now taking into account the invariance or invariance to within the sign of ω we get the following 12 cases

$$\left.\begin{array}{l|l|l}
 & \text{I}: r \to r & \text{II}: r \to -r \\
\underset{1}{\varepsilon} = 1 \;,\; \underset{2}{\varepsilon} = 1 \;: & \alpha = e^{i\varphi} \;\; ; \; \omega \to \omega & \beta = \lambda e^{i\varphi} \;\; ; \; \omega \to \omega \\
\underset{1}{\varepsilon} = 1 \;;\; \underset{2}{\varepsilon} = i \;: & \alpha = \begin{cases} e^{\zeta} \;\; ; \; \omega \to \omega \\ i e^{\zeta} \; ; \; \omega \to -\omega \end{cases} & \beta = \begin{cases} \lambda e^{\zeta} \;\; ; \; \omega \to \omega \\ i\lambda e^{\zeta} \; ; \; \omega \to -\omega \end{cases} \\
\underset{1}{\varepsilon} = i \;;\; \underset{2}{\varepsilon} = 1 \;: & \alpha = \begin{cases} e^{\zeta} \;\; ; \; \omega \to \omega \\ i e^{\zeta} \; ; \; \omega \to -\omega \end{cases} & \beta = \begin{cases} \lambda e^{\zeta} \;\; ; \; \omega \to -\omega \\ i\lambda e^{\zeta} \; ; \; \omega \to \omega \end{cases} \\
\underset{1}{\varepsilon} = i \;;\; \underset{2}{\varepsilon} = i \;: & \alpha = e^{i\varphi} \;\; ; \; \omega \to \omega & \beta = \lambda e^{i\varphi} \;\; ; \; \omega \to -\omega
\end{array}\right\} \quad (5.5)$$

with *real* parameters φ and ζ.

For $\underset{1}{\varepsilon} = \underset{2}{\varepsilon}$ the transformation in R_2 corresponding to $\alpha = e^{i\varphi}$ is a real rotation over 2φ in the direction $1 \to 2$:

$$\left.\begin{aligned} \underset{1}{i} &\to \underset{1}{i}\cos 2\varphi + \underset{2}{i}\sin 2\varphi \\ \underset{2}{i} &\to -\underset{1}{i}\sin 2\varphi + \underset{2}{i}\cos 2\varphi \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (5.6)$$

and the transformation corresponding to $\beta = \lambda e^{i\varphi}$ is a reflexotation [1])

$$\left.\begin{aligned} \underset{1}{i} &\to \underset{1}{i}\cos 2\varphi + \underset{2}{i}\sin 2\varphi \\ \underset{2}{i} &\to \underset{1}{i}\sin 2\varphi - \underset{2}{i}\cos 2\varphi \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (5.7)$$

that arises if (5.6) is followed by a reflexion at the new 1-axis. For both transformations we have (cf. (3.11))

$$\left.\begin{aligned} \underset{1}{i} &= \underset{1}{\varepsilon}\,(\lambda^{1}{}_{2} + 1/\lambda\,{}^{2}{}_{1}) \to \underset{1}{\varepsilon}\,(e^{2i\varphi}\,\lambda^{1}{}_{2} + e^{-2i\varphi}\,1/\lambda\,{}^{2}{}_{1}) \\ \underset{2}{i} &= i\,\underset{1}{\varepsilon}\,\lambda({}^{1}{}_{2} - 1/\lambda\,{}^{2}{}_{1}) \to i\,\underset{1}{\varepsilon}\,(e^{2i\varphi}\,\lambda^{1}{}_{2} - e^{-2i\varphi}\,1/\lambda\,{}^{2}{}_{1}), \end{aligned}\right\} \quad . \quad (5.8)$$

and that means that λ is transformed into $e^{2i\varphi}\lambda$. Hence the absolute value of λ is an invariant for the orthogonal group and we can use this to introduce in the first and fourth case of (5.2) as a *second invariant condition* that $|\lambda| = 1$. Then we get for the invariants ω and π

$$\left.\begin{aligned} \omega(1, 1) &= \pm(\bar{1}_1 + \bar{2}_2) \quad ; \quad \pi(1, 1) = \bar{\pi}^{-1} :: (\bar{1}_2 + \bar{2}_1) \\ \omega(i, i) &= \pm(\bar{1}_1 - \bar{2}_2) \quad ; \quad \pi(i, i) = \bar{\pi}^{-1} :: (\bar{1}_2 - \bar{2}_1) \end{aligned}\right\} \quad . \; . \quad (5.9)$$

and there exists always a pair of mutually perpendicular unitvectors $\underset{1}{i}$ and $\underset{2}{i}$ for which $\lambda = +1$ in (5.8).

For $\underset{2}{\varepsilon} = \pm i\,\underset{1}{\varepsilon}$ the transformation corresponding to $\alpha = e^{\zeta}$ is a rotation that is neither +-reflexional nor —-reflexional:

$$\left.\begin{aligned} \underset{1}{i} &\to \underset{1}{i}\cosh 2\zeta \mp \underset{2}{i}\sinh 2\zeta \\ \underset{2}{i} &\to \mp\underset{1}{i}\sinh 2\zeta + \underset{2}{i}\cosh 2\zeta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (5.10)$$

and the transformation corresponding to $\alpha = i\,e^{\zeta}$ is a rotation that is +- and —-reflexional:

$$\left.\begin{aligned} \underset{1}{i} &\to -\underset{1}{i}\cosh 2\zeta \pm \underset{2}{i}\sinh 2\zeta \\ \underset{2}{i} &\to \pm\underset{1}{i}\sinh 2\zeta - \underset{2}{i}\cosh 2\zeta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (5.11)$$

and that arises if (5.10) is followed by a reflexion at the origin.

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[1]) For $\nu = 1$ every reflexotation is a reflexion at some axis, e.g. (5.7) at the axis $\underset{1}{i}\cos\varphi + \underset{2}{i}\sin\varphi$.

For the same relation between $\underset{1}{\varepsilon}$ and $\underset{2}{\varepsilon}$ the transformation corresponding to $\beta = \lambda e^{\zeta}$ is a $\mp$-reflexional reflexotation

$$\left.\begin{array}{l} \underset{1}{i} \rightarrow \underset{1}{i} \quad \cosh 2\zeta \mp \underset{2}{i} \sinh 2\zeta \\ \underset{2}{i} \rightarrow \pm \underset{1}{i} \sinh 2\zeta - \underset{2}{i} \cosh 2\zeta \end{array}\right\} \quad \ldots \ldots \quad (5.12)$$

that arises if (5.10) is followed by a reflexion at the new 1-axis and the transformation corresponding to $\beta = i \lambda e^{\zeta}$ is a $\pm$-reflexional reflexotation

$$\left.\begin{array}{l} \underset{1}{i} \rightarrow -\underset{1}{i} \cosh\ 2\zeta \pm \underset{2}{i} \sinh 2\zeta \\ \underset{2}{i} \rightarrow \mp \underset{1}{i} \sinh 2\zeta + \underset{2}{i} \cosh 2\zeta \end{array}\right\} \quad \ldots \ldots \quad (5.13)$$

that arises if (5.10) is followed by a reflexion at the new 2-axis.

With all four transformations λ is transformed into $+ e^{\zeta}\lambda$ or $- e^{\zeta}\lambda$. Hence λ_a is an invariant for the orthogonal group and we can use this to introduce in the second and third case of (5.2) as a *second invariant condition* that $\lambda_a = 1$. Then we get for the invariants ω and π

$$\left.\begin{array}{l} \omega(1, i) = \pm\ (\bar{1}2 + \bar{2}1);\ \pi(1, i) = \bar{\pi}^{-1} :: (\bar{1}_1 + \bar{2}_2) \\ \omega(i, 1) = \pm i (\bar{1}2 - \bar{2}1);\ \pi(i, 1) = \bar{\pi}^{-1} :: (\bar{1}_1 - \bar{2}_2) \end{array}\right\} \quad (5.14)$$

and there exists always a pair of mutually perpendicular unitvectors $\underset{1}{i}$ and $\underset{2}{i}$ for which $\lambda = +1$ in (3.11).

§ 6. *Construction of C, K and Ω for ν > 1.*

According to the two invariant conditions introduced we use the representation (3.20) with $\lambda = 1$ in the definition (3.10) of p and q in each M_2. Then we try to write C, K and Ω as products of ν quantities from the ν different M_2's. *Note, that a general quantity of M_N can only be written as a sum of such products.*

In order to obtain that (4.15) be valid for $j = 1$ and $j = \nu + 1$, we have to take c as the first factor in C. But according to $c\,r\,c^{-1} = -'r$ (cf. (4.20) and $k\,p\,k^{-1} = -'p$; $k\,q\,k^{-1} = -'q$, we have to take k as the second factor in C in order to obtain that (4.15) be also valid for $j = 2$ and $j = \nu + 2$. According to $k\,r\,k^{-1} = -'r$ all r's in the first two columns of (3.20) have now changed their signs and we have to choose c as the third factor in C in order to obtain that (4.15) be also valid for $j = 3$ and $j = \nu + 3$. Proceeding in this way we get

$$C :: c \times k \times c \times k \times \ldots \quad (\nu \text{ factors}) \quad \ldots \quad (6.1)$$

and from $K = C R$ it follows that

$$K :: k \times c \times k \times c \times \ldots \quad (\nu \text{ factors}). \quad \ldots \quad (6.2)$$

From (5.9, 14) it follows that

$$\left.\begin{aligned}\omega(\eta_1,\eta_2)\,p\,\omega^{-1}(\eta_1,\eta_2)&=\eta_1^2\,{}'\bar{p}\\ \omega(\eta_1,\eta_2)\,q\,\omega^{-1}(\eta_1,\eta_2)&=\eta_2^2\,{}'\bar{q}\\ \omega(\eta_1,\eta_2)\,r\,\omega^{-1}(\eta_1,\eta_2)&\doteq\eta_1^2\,\eta_2^2\,{}'\bar{r}\end{aligned}\right\};\ \eta_1,\eta_2=1 \text{ or } i\,.\ .\ . \quad (6.3)$$

Hence, in order to make (4.22) valid for $j=1$ and $j=\nu+1$ in the representation (3.20) we have to take $\omega\,(\varepsilon_1, \varepsilon_{\nu+1})$ as the first factor in Ω. Then according to (6.3) the first column of (3.20) is transformed into

$$\left.\begin{matrix}\varepsilon_1^2\,{}'\bar{p} & \varepsilon_{\nu+1}^2\,{}'\bar{q}\\ \varepsilon_1^2\,\varepsilon_{\nu+1}^2\,{}'\bar{r} & \varepsilon_1^2\,\varepsilon_{\nu+1}^2\,{}'\bar{r}\\ \vdots & \vdots\\ \varepsilon_1^2\,\varepsilon_{\nu+1}^2\,{}'\bar{r} & \varepsilon_1^2\,\varepsilon_{\nu+1}^2\,{}'\bar{r}\end{matrix}\right\}.\ .\ .\ .\ .\ . \quad (6.4)$$

and from this it follows that we have to take $\omega\,(\varepsilon_1\,\varepsilon_{\nu+1}\,\varepsilon_2, \varepsilon_1\,\varepsilon_{\nu+1}\,\varepsilon_{\nu+2})$ as second factor in order to make (4.22) also valid for $j=2$ and $j=\nu+2$. Proceeding in this way we get $\omega\,(\varepsilon_1\,\varepsilon_2\,\varepsilon_{\nu+1}\,\varepsilon_{\nu+2}\,\varepsilon_3, \varepsilon_1\,\varepsilon_2\,\varepsilon_{\nu+1}\,\varepsilon_{\nu+2}\,\varepsilon_{\nu+3})$ for the third factor and so on till the ν-th factor $\omega\,(\varepsilon_1\ldots\varepsilon_\nu\,\varepsilon_{\nu+1}\ldots\varepsilon_{n-1}, \varepsilon_1\ldots\varepsilon_{\nu-1}\,\varepsilon_{\nu+1}\ldots\varepsilon_n)$.

From (5.9, 14) we see that $\omega\,(\eta_1,\eta_2)$ is definite only in the case $\eta_1=1$, $\eta_2=1$ and this has as a consequence that Ω is definite only if the signature of R_n is *positive definite* and indefinite in *all* other cases.

There is still another method for the computing of Ω. For the auxiliary quantity

$$\overset{0}{\Omega} \overset{\text{def}}{=} \omega(1,1)\times\ldots\times\omega(1,1) \qquad (\nu \text{ factors}),\ .\ .\ . \quad (6.5)$$

(this is the Ω for the positive definite case) we have obviously

$$\overset{0}{\Omega}\,\underset{j}{i}\,\overset{0}{\Omega}{}^{-1}=\varepsilon_j^2\,{}'i_j.\ .\ .\ .\ .\ .\ .\ .\ . \quad (6.6)$$

Hence, if $i_{\alpha_1}, \ldots, i_{\alpha_s}$ have the signature — and the others the signature +

$$\Omega::\left\{\begin{matrix}\overset{0}{\Omega}\,i_{\alpha_1}\ldots i_{\alpha_s} & \text{for } s \text{ even}\\ \overset{0}{\Omega}\,i_{\alpha_1}\ldots i_{\alpha_s}I & \text{for } s \text{ odd}\end{matrix}\right\}\ ^{2)}\ .\ .\ .\ . \quad (6.7)$$

This holds for every representation. In the representation (3.20) the matrix of $\overset{0}{\Omega}$ is the unity matrix and this can be expressed

$$\overset{0}{\Omega}\overset{*}{=}A.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (6.8)$$

---

2) This is the formula by which Cartan introduces his I and J and Weyl his B. Cf. § 4 at the end.

where the sign $\overset{*}{=}$ (as usual) means that the components of two quantities with different transformation laws happen to be equal for some definitely indicated coordinatesystem (here representation). Then we have the general formula

$$\Omega \overset{*}{::} \underset{\alpha_1}{i} \dots \underset{\alpha_s}{i}\, I^s \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6.9)$$

for the representation (3.20).

As we have seen C is either a tensor or a bivector. From $'C = \pm C$ and (4.20*a*) it follows that

$$C \underset{j_1 \dots p}{i} = \pm (-1)^{\binom{p}{2}} \underset{j_1 \dots j_p}{'i}\; 'C \quad . \; . \; . \; . \; . \; . \quad (6.10)$$

and from this and similar equations for K we get the following table where ( ) stands for symmetric, [ ] for alternating and where the heads of the columns still have to be accounted for:

$$\left.\begin{array}{lcc|lcc}
 & \binom{\nu+2}{2}\text{ odd} & \binom{\nu+2}{2}\text{ even} & & \binom{\nu+1}{2}\text{ even} & \binom{\nu+1}{2}\text{ odd} \\
C & (\ ) & [\ ] & K & (\ ) & [\ ] \\
C\underset{j}{i} & (\ ) & [\ ] & K\underset{j}{i} & [\ ] & (\ ) \\
C\underset{jk}{i} & [\ ] & (\ ) & K\underset{jk}{i} & [\ ] & (\ ) \\
C\underset{jkl}{i} & [\ ] & (\ ) & K\underset{jkl}{i} & (\ ) & [\ ] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
CR = K\begin{cases}\nu\text{ odd}\\ \nu\text{ even}\end{cases} & \begin{matrix}[\ ]\\ (\ )\end{matrix} & \begin{matrix}(\ )\\ [\ ]\end{matrix} & KR = C\begin{cases}\nu\text{ odd}\\ \nu\text{ even}\end{cases} & \begin{matrix}[\ ]\\ (\ )\end{matrix} & \begin{matrix}(\ )\\ [\ ]\end{matrix}
\end{array}\right\} \quad (6.11)$$

Now, the rank of C and K being N, every covariant spinor U_{AB} can be written in the form $U_{AB} = C_{AC}\, P^{C}_{.B}$ and also in the form $U_{AB} = K_{AC}\, Q^{C}_{.B}$ and P and Q can be expressed linearly into the $\underset{j}{i}$ and their products. Then U_{AB} can (as CP) be written as the sum of $1 + \binom{n}{1} + \binom{n}{4} + \binom{n}{5} + \dots$ independent bivectors (tensors) and $\binom{n}{2} + \binom{n}{3} + \binom{n}{6} + \binom{n}{7} + \dots$ independent tensors (bivectors) if the first (second) column in (6.11) is valid. Now U_{AB} is arbitrary and can always be written as the sum of $\binom{N}{2}$ independent bivectors and $\binom{N+1}{2}$ independent tensors, hence, either

$$\left.\begin{aligned}\binom{N}{2} &= 1 + \binom{n}{1} + \binom{n}{4} + \binom{n}{5} + \dots; \\ \binom{N+1}{2} &= \binom{n}{2} + \binom{n}{3} + \binom{n}{6} + \binom{n}{7} + \dots\end{aligned}\right\} \quad . \; . \quad (6.12)$$

or

$$\left.\begin{array}{l}\binom{N}{2} = \binom{n}{2} + \binom{n}{3} + \binom{n}{6} + \binom{n}{7} + \dots; \\ \binom{N+1}{2} = 1 + \binom{n}{1} + \binom{n}{4} + \binom{n}{5} + \dots \end{array}\right\} \quad . \quad . \quad . \quad (6.13)$$

But (6.12) holds only for $\binom{\nu+2}{2}$ odd and (6.13) for $\binom{\nu+2}{2}$ even. That accounts for the heads of the first two coluumns of (6.11). The heads of the other columns are accounted for in the same way starting from $K\,Q$.

Every quantity of valence 2 in $\mathfrak{S}_N$ can be decomposed in an invariant way into four parts by means of the two quantities

$$\begin{array}{l}\underset{1}{E} = {}^1/_2 (A+R) \\ \underset{2}{E} = {}^1/_2 (A-R)\end{array} \;;\ \text{in the representation (3.20):} \left\{\begin{array}{l}\underset{1}{E} = {}^{1}{}_{1} + \dots + {}^{N/2}{}_{N/2} \\ \underset{2}{E} = {}^{N/2+1}{}_{N/2+1} + \dots + {}^{N}{}_{N}\end{array}\right\} \quad (6.14)$$

e.g.

$$\underset{j}{i} = \underset{1}{E}\underset{j}{i}\underset{1}{E} + \underset{1}{E}\underset{j}{i}\underset{2}{E} + \underset{2}{E}\underset{j}{i}\underset{1}{E} + \underset{2}{E}\underset{j}{i}\underset{2}{E} \quad . \quad . \quad . \quad . \quad (6.15)$$

Using the representation (3.20) the matrix of every part has three quarters filled with zero's. A quantity whose matrix in the representation (3.20) has in the two quarters to the right and to the left of the main diagonal only zero's is called *reducible* (symbol: $\diagdown$). If the other quarters contain only zero's it is called *connecting* (symbol: $\diagup$). All quantities i, A, R, C, K, Ω, Π and their products are either reducible or connecting. This can be derived easily from the equations (3.8), (3.20), (6.1), (6.2), (6.5) and (6.7). The results are to be found in the final table (6.22).

From (4.26) it follows that $\Omega \underset{j_1 \dots j_p}{i}$ is hermitian for $\binom{p}{2}$ even and hermitian alternating for $\binom{p}{2}$ odd. In the table (6.22) this is indicated by ( ) and [ ] respectively.

In order to find out whether Π is positive or negative invertible we remark that a factor in Π with modulus 1 does not change this property and that

$$c \overset{*}{::} p \quad ; \quad k \overset{*}{::} i q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.16)$$

and accordingly

$$C \overset{=}{::} \left\{\begin{array}{l}\underset{1}{i} \dots \underset{\nu}{i} \text{ for } \nu \text{ odd} \\ \underset{1}{i} \dots \underset{\nu}{i} I \text{ for } \nu \text{ even}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6.17)$$

Hence, if among $i_1, \dots, i_\nu$ just $i_{\alpha_1}, \dots, i_{\alpha_{s_1}}$ have the —-signature and $i_{\beta_1}, \dots, i_{\beta_{\nu-}}$ the +-signature and in the same way among $i_{\nu+1}, \dots, i_n$ just $i_{\gamma_1}, \dots, i_{\gamma_{s_2}}$ have the — and $i_{\delta_1}, \dots, i_{\delta_{\nu-s_2}}$ the +-signature, we have

$$\Pi = {}'\Omega^{-1} C \mathbin{::} i_{\alpha_1} \dots i_{\alpha_{s_1}} i_{\gamma_1} \dots i_{\gamma_{s_2}} i_1 \dots i_\nu I^{s+\nu-1} \quad . \quad . \quad . \quad (6.18)$$

or

$$\Pi \overset{*}{=} \left\{ \begin{matrix} \varrho\, i_{\beta_1} \dots i_{\beta_{\nu-s_1}} i_{\gamma_1} \dots i_{\gamma_{s_1}} \text{ for } s+\nu \text{ odd} \\ \varrho\, i_{\alpha_1} \dots i_{\alpha_{s_1}} i_{\delta_1} \dots i_{\delta_{\nu-s_2}} \text{ for } s+\nu \text{ even} \end{matrix} \right\} . \quad . \quad . \quad (6.19)$$

and from this follows immediately

$$\bar{\Pi} = (-1)^{\binom{\nu-s}{2}} \Pi^{-1} \ \text{3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.20)$$

Hence (cf. § 4), if and only if $\binom{\nu-s}{2}$ is even there exists a coordinate system in $\mathfrak{S}_N$ such that $\Pi \overset{*}{=} A$ in this system. But according to

$$\Pi\, i_j\, \Pi^{-1} = \bar{i}_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.21)$$

this has as a consequence that real Majorana representations (cf. § 3) are possible if and only if $\binom{\nu-s}{2}$ is even.

From this it follows that representations with t_2 purely imaginary matrices for unitvectors in the —-region, t_1 purely imaginary matrices for unitvectors in the +-region and $n - t_1 - t_2$ real matrices, are possible if and only if $\binom{\nu - s - t_1 + t_2}{2}$ is even and that in this case I has a real matrix if $t_1 + t_2$ is even and a purely imaginary matrix if $t_1 + t_2$ is odd 4). This is the generalization of EDDINGTON's theorem mentioned in § 3. For $t_1 = s$, $t_2 = n - s$ we get the imaginary Majorana representations, possible only if $\binom{\nu + n - 3s}{2} = \binom{\nu - n + s}{2}$ is even.

---

3) Cf. VEBLEN and GIVENS, l.c. § 9, p. 10. We use here the definition $\binom{a}{2} = \frac{1}{2} a (a-1)$ also for negative values of a and for the values 0 and 1.

4) From this follows immediately the theorem of VEBLEN and GIVENS (l.c. § 9, p. 11) stating that if $2\nu + 1$ matrices of 2^ν rows and columns are at the same time square roots of $+A$ and mutually anticimmutative, and if among them u are purely imaginary and $2\nu + 1 - u$ real, this is only possible if $u \equiv \nu \pmod{4}$.

Here is a table of results:

	A	$\underset{j}{i}$	$\underset{jk}{i}$	…	I	I	$I\underset{j}{i}$	$I\underset{jk}{i}$	…	II
	╲	╱	╱	…	╲	╲	╱	╲		╲
	C	$C\underset{j}{i}$	$C\underset{jk}{i}$	…	CI	K	$K\underset{j}{i}$	$K\underset{jk}{i}$	…	KI
$\nu \equiv 1 \pmod 4$	(╱)	(╲)	[╱]	…	[╱]	[╱]	(╲)	(╱)	…	(╱)
$\nu \equiv 2 \pmod 4$	[╲]	[╱]	(╲)	…	[╲]	[╲]	(╱)	(╲)	…	[╲]
$\nu \equiv 3 \pmod 4$	[╱]	[╲]	(╱)	…	(╱)	(╱)	[╲]	[╱]	…	[╱]
$\nu \equiv 4 \pmod 4$	(╲)	(╱)	[╲]	…	(╲)	(╲)	[╱]	[╲]	…	(╲)
	$s =$ even					$s =$ odd				
	Ω	$\Omega\underset{j}{i}$	$\Omega\underset{jk}{i}$	…	ΩI	Ω	$\Omega\underset{j}{i}$	$\Omega\underset{jk}{i}$	…	ΩI
	(╲)	(╱)	[╲]	…	╲	(╱)	(╲)	[╱]	…	╱
	$s =$ even					$s =$ odd				
	Π	$\Pi\underset{j}{i}$	$\Pi\underset{jk}{i}$	…	ΠI	Π	$\Pi\underset{j}{i}$	$\Pi\underset{jk}{i}$		ΠI
$\nu \equiv 1 \pmod 4$	╱	╲	╱	…	╱	╲	╱	╲		╲
$\nu \equiv 2 \pmod 4$	╲	╱	╲		╲	╱	╲	╱		╱
$\nu \equiv 3 \pmod 4$	╱	╲	╱		╱	╲	╱	╲		╲
$\nu \equiv 3 \pmod 4$	╲	╱	╲		╲	╱	╲	╱		╱
	$\binom{\nu-s}{2}$ even: $\bar{\Pi} = +\Pi^{-1}$					$\binom{\nu-s}{2}$ odd: $\bar{\Pi} = -\Pi^{-1}$				

. . (6.22)

§ 7. *Non sufficiency of the invariance of R, C and Ω.*

We ask which homogeneous linear transformations in $\mathcal{S}_N$ with determinant ± 1 are fixed by the invariance of C and the invariance to within the sign of R and Ω (cf. the last table of § 4). Let the general transformation T with determinant ± 1 transform $\underset{j}{i}$ into

$$\underset{j'}{i} = T\underset{j}{i}T^{-1} = \underset{j}{a}^0 A + \sum_p^{1,\dots,n} \underset{j}{a}^{j_1\cdots j_p}\underset{j_1}{i}\cdots\underset{j_p}{i}\,. \quad . \quad . \quad . \qquad (7.1)$$

Now if Ω is invariant or invariant to within the sign we have according to (4.22), (4.36) with $\lambda = \pm 1$ and (7.1)

$$'\underset{j'}{\bar{i}} = {}'T^{-1}\,'\underset{j}{i}\,'\bar{T} = {}'T^{-1}\,\Omega\underset{j}{i}\Omega^{-1}\,'\bar{T} = \Omega T\underset{j}{i}T^{-1}\,\Omega^{-1} = \Omega\underset{j'}{i}\Omega^{-1} \qquad (7.2)$$

or

$$\underset{j}{\bar{a}}^0\,'\bar{A} + \sum_p^{1,\dots,n} \underset{j}{\bar{a}}^{j_1\cdots j_p}\,'\underset{i_p}{\bar{i}}\cdots'\underset{j_1}{\bar{i}} = \underset{j}{a}^0\,'\bar{A} + \sum_p^{1,\dots,n} \underset{j}{a}^{j_1\cdots j_p}\,'\underset{j_1}{\bar{i}}\cdots'\underset{j_p}{\bar{i}} \qquad (7.3)$$

and this is only possible if all coefficients a are real for $p = 0, 1, 4, 5, 8, 9, \dots$ and purely imaginary for $p = 2, 3, 6, 7, \dots$.

If C is invariant (or invariant to within the sign) we have according to (4.17), (4.42) and (7.1)

$$'\underset{j'}{i} = {}'T^{-1}\,'\underset{j}{i}\,'T = {}'T^{-1}\,C\underset{j}{i}C^{-1}\,'T = C\,T\underset{j}{i}T^{-1}\,C^{-1} = C\underset{j'}{i}C^{-1} \qquad (7.4)$$

or

$$\underset{j}{\alpha^0}\,'A+\sum_{p}^{1,\dots,n}\underset{j}{\alpha^{j_1\dots j_p}}\,'\underset{j_p}{i}\dots{}'\underset{j_1}{i}=\underset{j}{\alpha^0}\,'A+\sum_{p}^{1,\dots,n}\underset{j}{\alpha^{j_1\dots j_p}}\,'\underset{j_1}{i}\dots{}'\underset{j_p}{i} \tag{7.5}$$

and this is only possible if all coefficients α vanish for $p=2, 3, 6, 7, \dots$.

If R is invariant or invariant to within the sign we have according to (7.1), $R\underset{j}{i}=-\underset{j}{i}R$ and $TRT^{-1}=\pm R$

$$-\underset{j'}{i}=T(-\underset{j}{i})\,T^{-1}=TR\underset{j}{i}RT^{-1}=RT\underset{j}{i}T^{-1}R=R\underset{j'}{i}R \quad . \tag{7.6}$$

or

$$-\underset{j}{\alpha^0}A-\sum_{p}^{1,\dots,n}\underset{j}{\alpha^{j_1\dots j_p}}\underset{j_1}{i}\dots\underset{j_p}{i}=\underset{j}{\alpha^0}A+\sum_{p}^{1,\dots,n}(-1)^p\underset{j}{\alpha^{j_1\dots j_p}}\underset{j}{i}\dots\underset{j_p}{i} \tag{7.7}$$

and this is only possible if all coefficients α vanish for $p=0, 2, 4, 6, \dots$.

Taking all conditions together we may state that from the invariance or invariance to within the sign of Ω, C and R only follows that all coefficients α in (7.1) vanish except those for $p\equiv 1$ (mod. 4). But in order that the transformation (7.1) corresponds to an orthogonal transformation in R_n it is necessary and sufficient that all coefficients α in (7.1) vanish except those for $p=1$. Hence for $n\leqq 4$ the transformations in $\mathfrak{S}_N$ corresponding to orthogonal transformations in R_n are those that leave C invariant and R invariant to within the sign and among them the transformations also leaving Ω invariant to within the sign correspond to the real orthogonal transformations in R_n.

But for $n>4$ the invariance of Ω, C and R is not sufficient and there must exist other invariant quantities that are no comitants of Ω, C and R. Such an invariant is the spinor of valence 4

$$S^{A\;\;\dot{C}}_{\cdot\dot{B}\cdot D}={}^1\!/_n\sum_{j}^{1,\dots,n}\underset{j}{\varepsilon^2}\,\underset{j}{i}{}^{\dot{A}}_{\cdot B}\,\underset{j}{i}{}^{\dot{C}}_{\cdot D}\quad(\text{symbolic: } S={}^1\!/_n\sum_{j}\underset{j}{\varepsilon^2}\,\underset{j}{i}\circ\underset{j}{i}). \tag{7.8}$$

In fact, if $\underset{j'}{i}=T\underset{j}{i}T^{-1}$ is an orthogonal transformation in R_n, S is obviously invariant for T. In order to prove that every T leaving S invariant corresponds to an orthogonal transformation in R_n we write the transformation (7.1) in the form

$$\underset{j'}{i}=P_{j'}^{\;j}\underset{j}{i}+P_{j'}^{\;\mathfrak{a}}\,y_{\mathfrak{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{7.9}$$

symbolizing by the y all the products of the $\underset{j}{i}$'s and the quantity A. From (7.8) and the invariance of S it follows that

$$\left.\begin{aligned}\sum_{j}\underset{j}{\varepsilon^2}\,\underset{j}{i}\circ\underset{j}{i}=\sum_{j}\underset{j}{\varepsilon^2}\,\underset{j'}{i}\circ\underset{j'}{i}=\sum_{j'}\underset{j'}{\varepsilon^2}\,P_{j'}^{\;l}\underset{l}{i}\circ P_{j'}^{\;k}\underset{k}{i}&+\\ +\sum_{j'}\underset{j'}{\varepsilon^2}\,P_{j'}^{\;l}\underset{l}{i}\circ P_{j'}^{\;\mathfrak{a}}\,y_{\mathfrak{a}}&+\\ +\sum_{j'}\underset{j'}{\varepsilon^2}\,P_{j'}^{\;\mathfrak{a}}\,y_{\mathfrak{a}}\circ P_{j'}^{\;k}\underset{k}{i}&+\\ +\sum_{j'}\underset{j'}{\varepsilon^2}\,P_{j'}^{\;\mathfrak{a}}\,y_{\mathfrak{a}}\circ P_{j'}^{\;\mathfrak{b}}\,y_{\mathfrak{b}}&\end{aligned}\right\} \quad . \tag{7.10}$$

and this is possible if and only if

$$\left.\begin{array}{ll} a) & \sum\limits_{j'} \underset{j'}{\varepsilon^2} P_{j'}{}^{i}\, P_{j'}{}^{k} = \sum\limits_{j'} \underset{j}{\varepsilon^2}\, \delta^i_j\, \delta^k_j \\ b) & \sum\limits_{j'} \underset{j'}{\varepsilon^2} P_{j'}{}^{i}\, P_{j'}{}^{\mathfrak{a}} = 0 \\ c) & \sum\limits_{j'} \underset{j'}{\varepsilon^2} P_{j'}{}^{\mathfrak{a}}\, P_{j'}{}^{\mathfrak{b}} = 0. \end{array}\right\} \quad \ldots\ldots \quad (7.11)$$

From (7.11*a*) we see that the rank of $P_{j'}{}^{i}$ must be n. But then from (7.11*b*) it follows that $P_{j'}{}^{\mathfrak{a}} = 0$ and that, according to (7.9), $T \ldots T^{-1}$ represents an orthogonal transformation.

The hybrid spinor

$$Z^{\bar{A}\,.\,C}{}_{.\,\bar{B}\,.\,D} = {}^1\!/_n \sum_j \underset{j}{\varepsilon^2}\, \underset{j}{\bar{i}}{}^{\bar{A}}{}_{.\,\bar{B}}\; i^{C}{}_{.\,D}\text{; (symbolic: } Z = {}^1\!/_n \sum_j \underset{j}{\bar{\varepsilon}^2}\, \underset{j}{i} \circ \underset{j}{i}. \quad (7.12)$$

is invariant for all transformations in $\mathfrak{S}_N$ that correspond to real orthogonal transformations in R_n. Conversely only real orthogonal transformations in R_n leave S and Z invariant. This is proved in the same way as the corresponding property of S.

The spinor S can be looked upon as a linear transformation of spinors of valence 2 in three different ways:

$$P^{A}{}_{.\,B} \rightarrow S^{A\,.\,C}{}_{.\,B\,.\,D}\, P^{D}{}_{.\,C} \quad \ldots\ldots \quad (7.13)$$

$$Q_{BD} \rightarrow S^{A\,.\,C}{}_{.\,B\,.\,D}\, Q_{AC} \quad \ldots\ldots \quad (7.14)$$

$$O^{A}{}_{.\,D} \rightarrow S^{A\,.\,C}{}_{.\,B\,.\,D}\, O^{B}{}_{.\,C}. \quad \ldots\ldots \quad (7.15)$$

It is easily proved that the eigenvalues and eigenquantities in these three cases are

${}^{A}{}_{B}$ and ${}^{C}{}_{D}$			${}_{BD}$ and ^{AC}			${}^{A}{}_{D}$ and ${}^{C}{}_{B}$		
Number	Eigen-value	Eigen-quantity	Number	Eigen-value	Eigen-quantity	Number	Eigen-value	Eigen-quantity
n	${}^N\!/_n$	$\underset{j}{i}$	1	1	C	1	1	A
$N^2 - n$	0	$A, \underset{ij}{i}, \ldots, I$	n	$-\dfrac{n-2}{n}$	$C\underset{j}{i}$	n	$-\dfrac{n-2}{n}$	$\underset{j}{i}$
			$\binom{n}{2}$	$+\dfrac{n-4}{n}$	$C\underset{jk}{i}$	$\binom{n}{2}$	$+\dfrac{n-4}{n}$	$\underset{jk}{i}$
			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			1	-1	CI	1	-1	I
$\overline{N^2}$			$\overline{N^2}$			$\overline{N^2}$		

. (7.16)

The hybrid spinor Z can be looked upon as a linear transformation of hybrid spinors of valence 2 in two different ways

$$U_{\bar{B}D} \rightarrow Z^{\bar{A}\,.\,C}{}_{.\,\bar{B}\,.\,D}\, U_{\bar{A}C}\,. \quad \ldots\ldots \quad (7.17)$$

$$V^{\bar{A}}{}_{.\,D} \rightarrow Z^{\bar{A}\,.\,C}{}_{.\,\bar{B}\,.\,D}\, V^{\bar{B}}{}_{.\,C} \quad \ldots\ldots \quad (7.18)$$

The eigenvalues and eigenquantities for these cases are

$$\left.\begin{array}{ccc|ccc}
\multicolumn{3}{c|}{\overline{BD} \text{ and } \overline{AC}} & \multicolumn{3}{c}{{}^{\overline{A}}{}_{D} \text{ and } {}^{\overline{B}}{}_{C}} \\
\text{Number} & \text{Eigenvalue} & \text{Eigen-quantity} & \text{Number} & \text{Eigenvalue} & \text{Eigen-quantity} \\
1 & 1 & \Omega & 1 & 1 & \Pi \\
n & -\frac{n-2}{n} & \Omega\, i_{j} & n & -\frac{n-2}{n} & \Pi\, i_{j} \\
\binom{n}{2} & +\frac{n-4}{n} & \Omega\, i_{jk} & \binom{n}{2} & +\frac{n-4}{n} & \Pi\, i_{jk} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -1 & \Omega I & 1 & 1 & \Pi I \\
\overline{N^2} & & & \overline{N^2} & &
\end{array}\right\} \qquad (7.19)$$

From (7.16) and (7.19) we see once more that R, C and Ω are determined to within a scalar factor, and from these scalar factors we know already (§ 2 and § 4) that these factors can be fixed by normalizing conditions to within a factor ± 1, $e^{i\varphi}$ and ± 1 respectively.

(To be continued.)

Mathematics. — *Comments on Brouwer's Theorem on Essentially-negative predicates.* By D. VAN DANTZIG.

(Communicated at the meeting of October 29, 1949.)

1. In a recent note [1]) L. E. J. BROUWER constructed a real number ϱ, which he proved to be $\neq 0$ (i.e. he proved $\varrho = 0$ to be contradictory), without it being possible to prove either $\varrho > 0$ or $\varrho < 0$. This note, however, requires some comments because its validity depends 1° on the way in which some terms, not defined in the paper, are interpreted and 2° on some assumptions, partly of a psychological nature, which are not explicitly mentioned. Moreover the paper 1° is based on an idealistic philosophy, which certainly will not be accepted by all readers and thereby may invalidate it at least in appearance, and 2° is formulated in a terminology, fluctuating between "subjectivistic" and "objectivistic" [2]) statements — if only in appearance — which may make it difficult for some readers to form a judgment of their own on its validity.

In the present paper I shall try 1° to loosen the result from its philosophical origine, 2° to point out the assumptions underlying it and thereby 3° to make it better understandable to logicists. If some mathematicians and logicists — like myself — have been in doubt during some time whether the theorem were correct or not, I hope this paper may show them that, properly interpreted, it is. Moreover in § 1 I shall try to point out some peculiarities of Brouwer's terminology, and thereby, I hope, clarify it to some extent. In particular I shall try to show by some examples, how the paper can be "translated", either into a more consistently "subjectivistic", or into a more consistently "objectivistic" or "formal" terminology.

2. BROUWER defines the number ϱ in the following terms (translated from his Dutch text).

"Let α be a mathematical assertion [3]), which *cannot be "tested"*, i.e. no method is known [4]), in order to deduce either its absurdity [5]) or the absurdity of its absurdity [6])".

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[1]) L. E. J. BROUWER, Essentieel-negatieve eigenschappen, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 963—964 (1948); reprinted in Indagationes Mathematicae **10**, 322—323. Cf. also L. E. J. BROUWER, Consciousness, Philosophy and Mathematics, Proc. 10th International Congr. of Phil., 1948, vol. 2, p. 1235—1249.

[2]) I do not go here into the meaning of the terms 'objectivistic' and 'subjectivistic', corresponding with G. MANNOURY's distinction between "physicalistic" (after Carnap) and "introspective" terminology and consequently drop the quotation-marks further on.

[3]) As in intuitionistic mathematics a statement is only asserted if it has been proved, the term 'proposition' might express the author's intention more clearly than the term 'assertion' does.

"Then the creating subject can create in connexion with this assertion α an undefinitely proceeding sequence of rational numbers $a_1, a_2, a_3, \ldots$ according to the following instruction: As long as, during the choice of the a_n, neither the truth, nor the absurdity of α has become evident to the creating subject, each a_n is chosen $= 0$. As soon, however, as, between the choice of a_{r-1} and of a_r, the truth of α has become evident to the creating subject, a_r as well as $a_{r+\nu}$ for every natural ν is chosen $= 2^{-r}$. And as soon as, between the choice of a_{s-1} and of a_s the absurdity of α has become evident to the creating subject, a_s as well as $a_{s+\nu}$ for each natural ν is chosen $= -2^{-s}$.

"This undefinitely proceeding sequence $a_1, a_2, a_3, \ldots$ is positively convergent, hence it determines a real number ϱ."

§ 1. *Terminological remarks.*

3. In order not to deviate further from Brouwer's point of view than is strictly unavoidable, I shall admit here without criticism Brouwer's notion "unfinished infinite" (aftelbaar onaf"). [7])

4. It is not clear from Brouwer's paper, whether the term 'creative subject' is intended to denote:

1° the author himself,
2° an arbitrary human individual,
3° a human individual possessing some (which?) intellectual qualifications,
4° an „infinite" sequence of such individuals, successively performing the activities, ascribed to the creative subject,
5° a more or less definite *group* of human individuals, e.g. all mathematicians possessing some definite qualification,

or anything else.

Clearly the interpretations 1°—3° leave open the possibility that the creative subject dies after having chosen a finite number of numbers a_n so that the sequence cannot be continued and the real number ϱ remains undetermined. If the groups of assertions which have "become evident" to the subject between two successive definitions a_n, a_{n+1} consist of all theorems published in some well-defined journals or books, the interpretations 4° and 5° leave open the possibility that there is no unanimity between the mathematicians, as to whether a theorem has or has not been proved, in which case also the definition of the next a_n fails. Finally all these interpretations, apart from the concept of infinity itself, imply a semi-empirical hypothesis, viz. that there will "always" be a human being, willing and able to test the mathematical assertion a.

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[4]) BROUWER means: known to the creating subject.

[5]) The term 'absurd' as used by BROUWER, is meant as equivalent with "contrary to reason'; in Dutch it has not the meaning of 'nonsensical', it has in English.

[6]) "E.g. the assertion that 4 natural numbers $n > 2$, a, b and c exist, such that the relation $a^n + b^n = c^n$ holds, or the assertion that in the decimal development of π 10 successive digits occur, forming a sequence 0123456789." (Note by BROUWER.)

[7]) BROUWER, Over de grondslagen der wiskunde, thesis, 1907.

Perhaps the most consistently subjectivistic interpretation would be the first one, complemented by the admission that any individual reading the paper, or thinking about it, or having heard speaking about it, etc. may interpret it as denoting himself. As a subject does not die *subjectively*, this implies that any subject's concept of infinity corresponds with the concept of finiteness for any individual surviving him. This relativity of the meaning of the term 'infinite', which, in as far as I know, has never been admitted explicitly by BROUWER, was pointed out by G. MANNOURY and later by the present author to whom it seems to remain valid under any other interpretation of the term 'infinite' also. It also is not quite clear, what the author understands by a 'mathematical assertion'. I shall not go now into the possible interpretations of this term, nor into those of the expression 'has become evident to ...'.

5. Also Brouwer's use of the term 'absurd' needs some clarification, in particular as he intends it to be different from his use of the term 'not', although he is not always quite consistent in explicitly making this distinction [8]).

I think that the following interpretation of the term 'absurd' comes rather near to Brouwer's intention, if translated into objectivistic terminology:

> An assertion a is called 'absurd' by a subject S on a moment t, when he thinks he possesses on that moment a method, by which he can deduce a contradiction from any alleged demonstration of a whatsoever.

Here the term 'a contradiction' will probably be considered by BROUWER as being intuitively clear, by formalists as meaning 'a statement of the form '$A \wedge \neg A$' ', and by some other mathematicians and logicians as needing further clarification, and perhaps not being definable without recurrence to the notion of absurdity. Moreover BROUWER probably would omit the words 'he thinks' and would probably not accept an interpretation of 'any ... whatsoever' as 'any ... S can on the moment t imagine', which I should accept. Also the term 'deduce' would perhaps be interpreted by BROUWER in a more "absolutistic" sense than by me.

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[8]) E.g. '... then ϱ could not be < 0' is intended as '... then $\varrho < 0$ were absurd'; '... which is not the case' is intended as '... which contradicts the assumptions', i.e. as '... which, the assumptions supposed to be true, is absurd'; '... $\varrho > 0$ does not hold' is meant as '$\varrho > 0$ has not yet been proved'. The term 'contradictory' in '... $\varrho = 0$ is contradictory' is used as a synonym of 'absurd'.

In any case it seems impossible to define Brouwer's use of the term 'absurd' without using another negative term (like 'contradiction'), in other words it seems to be unavoidable to consider Brouwer's general form of the negation as an "entité primitive", irreducible to affirmative concepts. Therefore Brouwer's theorem can not invalidate the attempts of other mathematicians (in particular G. F. C. GRISS, Mrs. P. DESTOUCHES–FÉVRIER and the present author) to do without the concept of negation. On the contrary, it shows again, how unclear the concept of 'absurdity' is, and makes its avoidance appear more desirable. On the other hand I might point out that I consider affirmative mathematics as an interesting, but not as the only interesting, and still less as the only "justified" part of mathematics [9]).

6. Finally I might point to Brouwer's hesitating between a rather "subjectivistic" and a more "objectivistic" terminology. Sometimes he refers his statements explicitly to a "creative subject" (here denoted by S), sometimes he omits this reference, although (or because) it can readily be added, sometimes, however, he formulates his sentences as if they did not refer to S at all. Although I believe that BROUWER intends the reference to S in the latter places also, a translation, either (A) into a more consistently "subjectivistic" or (B) into a more consistently "objectivistic" or "formal" terminology — which latter terminology, of course, does not correspond with Brouwer's views — might make the paper better understandable to many readers. I give some examples, where the consistency still is far from complete. The italics are mine.

1° '... a *mathematical* assertion'. Translations: A. '... an assertion to which S applies the predicate 'mathematical', such that S believes he will never be in doubt, whether it has or has not been proved by him — or to his satisfaction — nor whether it has or has not been "reduced to absurdity" ...'. B. 'a formula occurring in a formal system S — called 'mathematical' within some meta-system M —, formed in accordance with some rules, formulated in a meta-system of S — the "syntaxis" of S'.

2° '... which *cannot* be tested' Translation: A. '... by S on the moment under consideration'; B. '... such that the applications of a system of operations — called 'testing' —, formulated in a meta-system M of S — called the 'semantics of S' — neither results in a formula of the form $T[\neg\neg\alpha]$, nor in a formula of the form $T[\neg\alpha]$ — where $T[\beta]$ is a predicate in M, defined for elements of S as variables, subject to some rules formulated in M, — and called 'β is true' —'.

3° '... no method is *known*'. To add: 'to S on the moment t'.

4° '*S can* ...'. Translation: either 'S expects he can ...' or 'I — BROUWER — expect that S can ...'.

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[9]) Cf. D. VAN DANTZIG, Mathématique stable et mathématique affirmative, Congr. Intern. de Philos. des Sciences, Paris 1949; —, On the principles of affirmative and intuitionistic mathematics (1941), Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 918—929; 1092—1103 (1947).

5° 'This ... sequence ... *is* ... convergent ...'. To read: '... has been proved to S's — or to mine, Brouwer's — satisfaction to be convergent'. Translation with reference to an imagined super-intelligence — contrary to the general trend of Brouwer's ideas —: '... will be found by "posterity" to be ...' or '... is known by "God" to be ...'.

6° 'If ... $\varrho > 0$ ... did *hold* ...': 'If ... $\varrho > 0$ ... had been proved by S on or before the moment t' ...'. Here t' is *not* meant to be necessarily the same moment as t; it may be any later moment.

I hope that after these examples many a reader will find it not very difficult, but rather illuminating and facilitating to grasp Brouwer's intentions, to translate the paper either into a "subjectivistic" or into an "objectivistic" — or "formal" — terminology.

Resuming the content of this paragraph I might state that a consistently subjectivistic — which probably is the one most akin to Brouwer's intentions — interpretation of Brouwer's paper implies that the terms 'infinite', 'evident', 'absurd' (and other ones) should be used in a "subjectively absolute" way, which is untenable from an objectivistic point of view, hence also as soon as the subject recognises any other subject different from himself. The validity of Brouwer's theorem, however, is *not* restricted to this subjectivistic interpretation, as I shall show in the second part of this paper.

§ 2. *Interpretation of Brouwer's theorem.*

7. Although it is not completely clear, what the term 'creative subject' is intended to mean (cf. § 1, 4), it is clear that the *task* of this subject with regard to the present theorem is 1° to create the numbers a_n according to the definition, a task which is purely formal and which it can have performed by an appropriate machine, 2° in between two successive definitions of numbers a_{n-1}, a_n to prove — or controll proofs of — mathematical theorems, in particular of the proposition α. Let ω_n be the set of deductions it has performed or controlled between the definitions of a_{n-1} and a_n.

After the deductions ω_n and the determination γ_n of a_n the subject gives a discussion δ_n, based on the preceding performances, regarding the real number ϱ. We denote by δ_0 Brouwer's definition of ϱ, by σ_n the conjunction of the sets of deductions $\omega_1, \ldots, \omega_n$, by Ω_n the set $\omega_{n+1}, \omega_{n+2}, \ldots$, and by $\succ$ the predicate 'contains a deduction of'.

Putting $a_0 = 0$ the definition γ_n can then be formulated as follows: 10)

if $a_{n-1} \neq 0$ then $a_n = a_{n-1}$;
if $a_{n-1} = 0$ then 11) '$\omega_n \succ \alpha$' $\equiv$ '$a_n = 2^{-n}$'
and '$\omega_n \succ \neg\, \alpha$' $\equiv$ '$a_n = -2^{-n}$'
and '$\neg$ '$\omega_n \succ \alpha$' $\wedge \neg$ '$\omega_n \succ \neg\, \alpha$'' $\equiv$ '$a_n = 0$'.

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10) It does not imply a restriction, if α is replaced by its double negation; then α is "stable" and everywhere $\neg\neg\, \alpha$ can be replaced by α.

It is tacitly assumed, that 1° the sequence of sets of deductions $\omega_1, \omega_2, \ldots$ does not contain a contradiction — e.g. that the subject is sufficiently educated in mathematics not to make errors —, 2° that for any n there is no doubt [12]), whether ω_n does or does not contain a deduction either of the assertion α or of its negation $\neg\, \alpha$.

Under these conditions a_n is uniquely determined by a_{n-1} and ω_n.

The question is left open, which further conditions are necessary in order to exclude the possibility that α by being dependent upon δ_0 becomes self-contradictory (e.g. if $\alpha \equiv$ ‘$\varrho < 0$’) or by dependence upon the ω_n might become undecidable (e.g. $\alpha \equiv \bigcup\limits_{1}^{\infty} \omega_n$).

It is a natural consequence of the principles of intuitionistic mathematics to require that each ω_n be a *finite* set of deductions, each application of complete or transfinite induction being counted as one deduction.

In connection with existing theories of deducibility, it is of importance to remark that ‘$\omega_n \succ \alpha$’ does not mean that α is deduc*ible* from ω_n, but that a deduction of α *actually occurs* in ω_n (“α becomes evident to the subject”). In order to get a clear distinction between σ_n and the subsequent deductions δ_n it is therefore desirable to require that any deduction occurring in δ_n which does not depend on δ_0 occurs in σ_n already.

8. Hence ϱ is uniquely determined *as a function of* $\omega_1, \omega_2, \ldots$. Now, the most natural interpretation of Brouwer's statement that ‘$\varrho = 0$’ is absurd, would be that this were so, *whatever* $\omega_1, \omega_2, \ldots$ were, i.e. *identically in the* ω_n. This, however, is not true. For if, e.g. all ω_n are empty, then *for these* ω_n $\varrho(\omega_1, \omega_2, \ldots) = 0$. This is the case, if the subject sleeps between every two successive γ_n, or also if he does not occupy himself with the parts of mathematics to which α belongs, etc.

On several occasions [13]) BROUWER has explicitly stated that the freedom of a subject to choose his own activity, according to his views, implies at any moment the freedom to restrict this freedom further on. Unless this is not admitted for its activity in general, which seems somewhat inconsistent, this decidedly leaves open the possibility that S decides on some moment nevermore to occupy himself with the assertion α, but nevertheless to go on with the construction of the numbers a_n. Under this decision Ω_n becomes

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[11]) Single inverted and upright upper commas are used as opening and closing brackets respectively. Pairs of commas have been omitted, when no misunderstanding was expected.

[12]) This consistent neglect of the possibility of doubt as to whether a construction has or has not been performed by the subject, is the most important formal element in intuitionistic mathematics.

[13]) E.g. L. E. J. BROUWER, De non-aequivalentie van de constructieve en de negatieve orderelatie in het continuum, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **52**, 122—124 (1949) (Indag. Math. **11**, 37—39 (1949): “... which have retained their complete freedom of continuation, inclusive their freedom of later restriction of freedom” (translated from BROUWER's Dutch text). Here, however, only choices of numbers for the construction of points of the continuum, not of other activities, are considered.

empty and $\varrho = 0$ becomes true instead of contradictory. Brouwer's theorem can therefore not be interpreted as: for every choice of the ω_n it is absurd that $\varrho = 0$, but only as: it is absurd that $\varrho = 0$ for every choice of the ω_n. This distinction is deciding for the validity of Brouwer's theorem.

Brouwer's proof can be formulated thus:

If $\delta_n \succ ‘\varrho \leqq 0’$, then not only for no m $\omega_m \succ \alpha$, but this must follow already from σ_n, i.e. 14) $\sigma_n \succ ‘\forall_m \neg ‘\omega_m \succ \alpha’’$. Herefrom BROUWER concludes $\sigma_n \succ \neg \alpha$ (cf. 9). Hence:

$$‘\delta_n \succ ‘\varrho \leqq 0’’ \supset ‘\sigma_n \succ \neg \alpha’ \quad \ldots\ldots \quad (1)$$

and in the same way

$$‘\delta_n \succ ‘\varrho \geqq 0’’ \supset ‘\sigma_n \succ \neg\neg \alpha’. \quad \ldots\ldots \quad (2)$$

Hence

$$‘\delta_n \succ ‘\varrho = 0’’ \supset ‘\sigma_n \succ ‘\neg \alpha \wedge \neg\neg \alpha’’ \quad \ldots\ldots \quad (3)$$

hence

$$\neg ‘\delta_n \succ ‘\varrho = 0’’ \quad \ldots\ldots \quad (4)$$

as otherwise σ_n would contain a contradiction.

9. The transition from $\neg ‘\Omega_n \succ \alpha’$ to $\neg \alpha$ requires that the former statement be true *for any* Ω_n. Then it is a consequence of Brouwer's concept of truth. An assertion α is only then admitted to be true, if a demonstration of it exists: $\exists_\Omega ‘\Omega \succ \alpha’$. Hence if $\forall_\Omega \neg ‘\Omega \succ \alpha’$, i.e. if a deduction is obtained, which from *any* alleged deduction Ω of α deduces a contradiction, α can not be true, i.e. $\neg \alpha$ is true.

For the correct understanding of this conclusion it is of importance to remark that *in intuitionistic mathematics an assertion α can not be — i.e. be proven to be — undecidable*. If it were, then we should have a deduction of a contradiction from any alleged proof of $‘\alpha \vee \neg \alpha’$, in particular also of α, from which $\neg \alpha$ would follow. In the same way $\neg\neg \alpha$ would follow, so that an alleged proof of the undecidability of α leads to the contradiction $\neg \alpha \wedge \neg\neg \alpha$. Formally, as A. HEYTING showed, the classical implication $‘\neg ‘\alpha \vee \beta’’ \supset ‘\neg \alpha \wedge \neg \beta’$ remains valid in intuitionistic logic, (contrary to $‘\neg ‘\alpha \wedge \beta’’ \supset ‘\neg \alpha \vee \neg \beta’$), from which the contradiction follows when β is replaced by $\neg \alpha$.

The non-existence of intuitionistically undecidable propositions together with the existence of propositions which are undecidable within a definite formal system is difficult to understand by means of the formal distinction between a system and its semantic meta-system. The paper under discussion, as well as Brouwer's previous paper on the continuity of everywhere defined functions, based on the theory of demonstration, make it

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14) When using the quantifiers $\forall$ und $\exists$, we omit for brevity the statements that m and n denote natural numbers, Ω subsets of Σ, and $\omega_1, \omega_2, \ldots$ finite subsets of Ω.

appear dubious whether it will be possible on the long run to maintain the strict distinction between formal systems and their semantic meta-systems against Brouwer's opinion that this is not possible.

Hence Brouwer's conclusion requires that $\varrho \leqq 0$ is proved in δ_n, whatever Ω_n be. Indicating explicitly the dependence of ϱ upon Ω_n by writing $\varrho_n(\Omega_n)$ instead of $\varrho = \varrho(\omega_1, \omega_2, \ldots)$, with $\Omega_n = (\omega_{n+1}, \ldots)$, we have therefore:

$$‘\delta_n > \forall_\Omega ‘\varrho_n(\Omega) \leqq 0’’ \supset ‘\sigma_n > ‘\forall_\Omega \neg ‘\Omega > \alpha’’’ \supset ‘\sigma_n > \neg \alpha’$$

Hence Brouwer's statement that $\varrho = 0$ is contradictory must be meant in the sense:

$$\forall_n \neg ‘\delta_n > \forall_\Omega ‘\varrho_n(\Omega) = 0’’$$

i.e. for no n can $\varrho_n(\Omega) = 0$ (*identically in* Ω) be proven from $\omega_1, \ldots, \omega_n$.

Writing this for a moment shortly as $\neg \forall_\Omega ‘\varrho_n(\Omega) = 0’$, the decisive point mentioned above consists in distinguishing this assertion from $\forall_\Omega \neg ‘\varrho_n(\Omega) = 0’$, which is not true. As long as the dependence of ϱ upon $\omega_1, \omega_2, \ldots$ is not explicitly mentioned, both statements are symbolised by $\neg ‘\varrho = 0’$ and the distinction remains impossible.

Further Brouwer argues: if $\neg ‘\varrho_n(\Omega) \geqq 0’$ 15) for some Ω, then this implies that $\neg \alpha$ is true. Hence, if $\delta_n > \exists_\Omega \neg ‘\varrho_n(\Omega) \geqq 0’$, then the truth of $\neg \alpha$ must have been concluded from σ_n already:

$$‘\delta_n > \exists_\Omega \neg ‘\varrho_n(\Omega) \geqq 0’’ \supset ‘\sigma_n > \neg \alpha’$$

and in the same way:

$$‘\delta_n > \exists_\Omega \neg ‘\varrho_n(\Omega) \leqq 0’’ \supset ‘\sigma_n > \neg\neg \alpha’.$$

Hence in this case we can draw in δ_n not only the conclusion $\exists_\Omega \neg ‘\varrho_n(\Omega) \geqq 0’$ (or $\leqq 0$), but even

$$\forall_{\omega_{n+1}} \forall_{\omega_{n+2}} \ldots ‘\varrho_n(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \leqq -2^{-n}’$$

(or $\geqq +2^{-n}$ respectively).

10. We can eliminate the concept of 'creating subject' altogether, by replacing this subject by the produce of its activity. Brouwer's theorem can then be given the following form, which, I think, is acceptable to logicists as well as — if the restrictions 'according to ...' in 1° and 2° are omitted — to intuitionists.

Hypotheses.

1° A is a set of formulae, representing mathematical assertions, according to the rules of a given syntactical meta-system.

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15) BROUWER's notation for $\neg ‘x \geqq 0’$ is: $x < 0$.

2° Σ is a set of deductions of formulae belonging to A, according to the rules of a given semantical metasystem.

3° Σ is free from contradiction.

4° $\omega_1, \omega_2, \omega_3, \ldots$ are finite sets of deductions belonging to Σ, (each application of complete or transfinite induction being counted as one deduction).

5° $\sigma_n = \bigcup\limits_{1}^{n} \omega_i$.

6° If $\text{‘}\vartheta \succ \beta\text{’}$ denotes for any $\beta \in A$ and any $\vartheta \subset \Omega$, that a deduction of β occurs in ϑ, then $\text{‘}\vartheta \succ \beta\text{’} \vee \neg\, \text{‘}\vartheta \succ \beta\text{’}$.

7° $\delta_0\,(\alpha; \omega_1, \omega_2, \ldots)$ (shortly δ_0) for any $\alpha \in A$, and any sequence $\omega_n \subset \Omega$ is the conjunction of the following definitions (1)—(5) *not* contained in A:

$$\text{‘}\mathbf{a}_0\,(\alpha) = 0\text{’} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$\text{‘}\forall_{n \geqq 1}\, \text{“}\neg\, \text{‘}\sigma_n \succ \alpha\text{’} \wedge \neg\, \text{‘}\sigma_n \succ \neg\, \alpha\text{’’} \supset \text{‘}\mathbf{a}_n\,(\alpha; \omega_1, \ldots, \omega_n) = 0\text{’’’} \;. \quad (2)$$

$$\text{‘‘‘}\sigma_n \succ \alpha\text{’} \wedge \text{‘}m = \min_{\sigma_k \succ \neg\neg\, \alpha} k\text{’’} \supset \text{‘}\mathbf{a}_n\,(\alpha; \omega_1, \ldots, \omega_n) = 2^{-m}\text{’’} \;. \quad (3)$$

$$\text{‘‘‘}\sigma_n \succ \neg\, \alpha\text{’} \wedge \text{‘}m = \min_{\sigma_k \succ \neg\, \alpha} k\text{’’} \supset \text{‘}\mathbf{a}_n\,(\alpha; \omega_1, \ldots, \omega_n) = -2^{-m}\text{’’} \quad (4)$$

$$\text{‘}\varrho\,(\alpha; \omega_1, \omega_2, \ldots) = \lim_{n \to \infty} \mathbf{a}_n\,(\alpha; \omega_1, \ldots, \omega_n)\text{’}. \quad . \; . \; . \; . \quad (5)$$

8° δ_n is a finite set of deductions which can be drawn from δ_0 and σ_n 16).

9° If a deduction of an assertion not depending on δ_0 is contained in δ_n then it is contained in σ_n already.

Then Brouwer's theorem is aequivalent with the following statement: For all natural $n \geqq 1$:

1° $$\neg\, \forall^{\Omega}_{\omega_{n+1}}\, \forall^{\Omega}_{\omega_{n+2}} \ldots \text{‘}\varrho\,(\alpha; \omega_1, \omega_2, \ldots) = 0\text{’}$$

2° $$\text{‘}\sigma_n \succ \neg\, \alpha\text{’} \equiv \delta_n \succ \text{‘}\exists^{\Omega}_{\omega_{n+1}}\, \exists^{\Omega}_{\omega_{n+2}} \ldots \neg\, \text{‘}\varrho\,(\alpha; \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \geqq 0\text{’’} \equiv$$

$$\equiv \delta_n \succ \text{‘}\forall^{\Omega}_{\omega_{n+1}}\, \forall^{\Omega}_{\omega_{n+2}} \ldots \text{‘}\varrho\,(\alpha; \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \leqq -2^{-n}\text{’’}$$

3° $$\text{‘}\sigma_n \succ \neg\neg\, \alpha\text{’} \equiv \delta_n \succ \text{‘}\exists^{\Omega}_{\omega_{n+1}}\, \exists^{\Omega}_{\omega_{n+2}} \ldots \neg\, \text{‘}\varrho\,(\alpha; \omega_1, \ldots \omega_n, \omega_{n+1}, \ldots) \leqq 0\text{’’} \equiv$$

$$\equiv \delta_n \succ \text{‘}\forall^{\Omega}_{\omega_{n+1}}\, \forall^{\Omega}_{\omega_{n+2}} \ldots \text{‘}\varrho\,(\alpha; \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \geqq +2^{-n}\text{’’}$$

With the corresponding interpretations Brouwer's demonstration remains valid.

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16) Formally the specification of the deductions which are admitted in δ_n needs some more precision than is given in 8° and 9°.

Aerodynamics. — *Solutions of the equations for the non-uniform propagation of a very strong shock wave.* I. By W. P. ROBBERTSE and J. M. BURGERS. (Mededeling no. 63 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of October 29, 1949.)

1. In the thesis of the first author "Oor die verdigtings- en verdunningsverskijnsels in 'n gas veroorsaak deur die stoot van 'n suier met 'n baie hoë snelheid" [1], it has been pointed out that solutions in finite terms for the non-uniform propagation of a very strong shock wave can be obtained for certain special distributions of the density of the gas. These solutions were constructed on the assumption that the specific expansion of the individual layers of the gas, as described by a function $\Phi = \partial\varphi/\partial s$, should be represented by an expression of the form:

$$\Phi = \frac{\text{(function of } t)}{\text{(function of } s)}.$$

Two examples of such a solution have been considered in § 4 (p. 19—25) of the thesis.

In the following lines a more general solution will be given. These solutions are found to correspond to cases where the shock wave propagates itself with increasing velocity through a gas with decreasing density. Since the increase of the velocity of propagation entails an increase of the temperature of the compressed gas, the results may be of importance for certain astrophysical problems.

The method applied in the following lines is based on the same ideas as those developed in the thesis and also in an earlier paper by the second author [2]. Recently it was found by the second author that the rather complicated boundary condition to be satisfied at the shock wave could be replaced by a simpler one, leading to a more convenient procedure for solving the equations. In order to explain the new condition, it is necessary shortly to recapitulate some of the points of view occurring in the original exposition of the method.

The flow of the gas is represented in an x, t-plane (fig. 1). The paths of the various layers of the gas are curves $x = \varphi(s; t)$, the value of s being a constant for each individual layer. The way in which the values of s are chosen will be indicated in section 2. Since we consider the motion initiated by the passage of the compression wave front (the shock wave), these paths start from the trace of the shock wave in the x, t-plane, which itself is described by a curve $x = X(t)$.

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[1] W. P. ROBBERTSE, Thesis Delft 1948 (Meded. no. 57 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft).

[2] J. M. BURGERS, Aerodynamical problems connected with the motion of a cloud of gas emitted by Nova Persei, these Proceedings **50**, 262 (1947).

We assume the state of motion of the gas before it suffered the effect of the shock wave, to be known. Hence in the domain of the x, t-plane below (and to the right of) the curve $x = X(t)$, the velocity u_0, the density ϱ_0 and the pressure p_0 are given functions of x and t. It may be supposed that the gas is under the influence of gravity, acting in the x-direction with constant intensity $-g$ per unit mass [3].

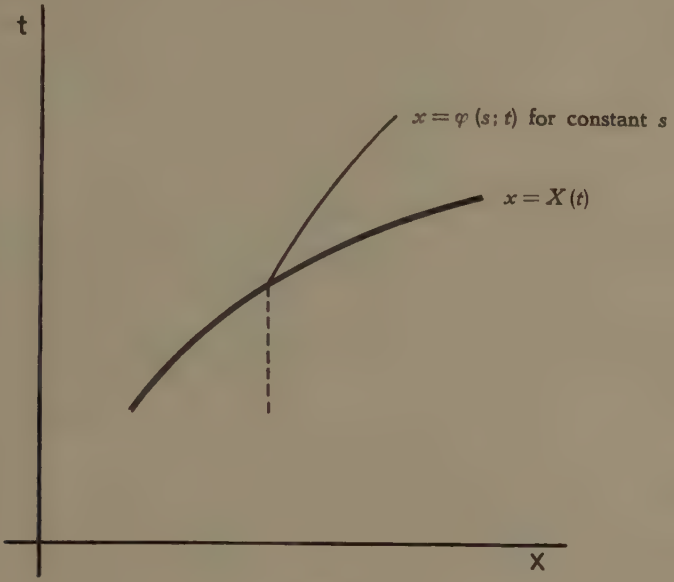

Fig. 1.

The law of propagation of the shock wave will not only depend on the original state of the gas but also on phenomena occurring behind the shock wave, since there may be expansion waves travelling through the gas and overtaking the shock wave. The function $X(t)$ consequently is an unknown of the problem. Nevertheless, in order to arrive at a convenient formulation of certain conditions, it is useful provisionally to assume that $X(t)$ is given [4]. — The treatment will be restricted to the case where $X(t)$ and its derivatives are *continuous*; cases in which the shock wave is overtaken by another shock wave (or where it should meet another shock wave) will not be considered.

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[3] The influence of gravity will be considered as small compared with the main phenomenon; when necessary g will therefore be treated as a very small quantity.

The method of solution developed in this paper can also be applied to the case where g is a linear function of x ($g = A - Bx$, A and B being constants).

[4] Compare R. COURANT and K. O. FRIEDRICHS, Supersonic Flow and Shock Waves (Interscience Publishers, New York, 1948), p. 171.

2. The velocity of propagation of the shock wave, at the point $x = X$, is given by:

$$\xi = dX/dt. \quad \ldots \ldots \ldots \quad (1)$$

Let u_0, c_0 represent the velocity of motion of an element of the gas and the velocity of sound in that element in the original state of the gas, at the point $x = X$, at the instant when this point is reached by the shock wave. The velocity u_1 of the gas immediately behind the shock wave (immediately after the passage of the wave front) is then determined by [5]:

$$u_1 = u_0 + \frac{2\,(\xi - u_0)}{k+1} - \frac{2\,c_0^2}{(k+1)\,(\xi - u_0)}. \quad \ldots \ldots \quad (2)$$

At the same instant the density ϱ_1 and the pressure p_1 of the gas are given by:

$$\varrho_1 = \varrho_0 \frac{\xi - u_0}{\xi - u_1} \quad ; \quad p_1 = p_0 + \varrho_0\,(\xi - u_0)\,(u_1 - u_0). \quad \ldots \quad (3)$$

At the shock wave the function $\varphi(s; t)$ must satisfy the condition:

$$\left.\begin{matrix}\text{velocity of the element}\\ \text{of volume of the gas}\end{matrix}\right\} = \frac{\partial\varphi}{\partial t} = u_1. \quad \ldots \ldots \quad (4)$$

We write:

$$\partial\varphi/\partial s = \Phi \quad \ldots \ldots \ldots \quad (5)$$

so that Φ is a measure for the "specific expansion" of the element of volume. It is convenient to subject the function φ to the further condition:

$$\partial\varphi/\partial s = 1 \text{ at the shock wave.} \quad \ldots \ldots \quad (6)$$

This condition fixes the way in which the parameter s, labelling the paths of the various individual layers, is counted. It does not introduce a physical relation into the problem, but implies that Φ measures the specific expansion from the state of the gas immediately after the passage of the shock wave. Hence, assuming that after this passage the relation between density and pressure is governed by POISSON's law, for every element of volume, we have:

$$\varrho = \varrho_1\,\Phi^{-1} \quad ; \quad p = p_1\,\Phi^{-k}. \quad \ldots \ldots \quad (7)$$

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[5] The states of the gas in front and at the back of the shock wave are connected by the well known equations:

$$\varrho_0\,(\xi - u_0) = \varrho_1\,(\xi - u_1)$$

$$p_0 + \varrho_0\,(\xi - u_0)^2 = p_1 + \varrho_1\,(\xi - u_1)^2$$

$$\frac{k\,p_0}{(k-1)\,\varrho_0} + \frac{(\xi - u_0)^2}{2} = \frac{k\,p_1}{(k-1)\,\varrho_1} + \frac{(\xi - u_1)^2}{2}.$$

Elimination of p_1 and ϱ_1 leads to eq. (2) of the text, while eqs. (3) will be immediately evident.

Along the trace of the shock wave the following relations hold:

(*a*) $$d\varphi = (\partial\varphi/\partial s)\, ds + (\partial\varphi/\partial t)\, dt = ds + u_1\, dt;$$

(*b*) $$d\varphi = dX = \xi\, dt.$$

Hence:

$$ds = (\xi - u_1)\, dt\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and [6]):

$$S' = ds/dt = \xi - u_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

When X is considered as a known function of t, and u_1 is obtained from (2), we can use eq. (9) to find the values of s along the trace of the shock wave. In this way the labelling of the paths of the individual layers of the gas will be completely determined.

Now suppose that the function $\varphi(s; t)$ were known. The trace of the shock wave could then be found from the condition $\partial\varphi/\partial s = 1$ along this trace. From this equation we deduce:

$$S' = ds/dt = -\frac{\partial^2\varphi/\partial s\partial t}{\partial^2\varphi/\partial s^2}.$$

With the aid of (2) and (9) it thus is possible to calculate ξ and u_1. The latter quantity on the other hand is given by (4). It follows that the function $\varphi(s; t)$ must satisfy a certain condition at the shock wave, which is obtained by the elimination of ξ and u_1 from these equations. This condition replaces the condition introduced in BURGERS' original paper and applied in ROBBERTSE's thesis, which referred to certain derivatives of the function Φ. It is not necessary at this stage to give the new condition explicitly, since it will occur in the next section in a simplified form (eq. 17).

3. In the following sections we restrict ourselves to the case where the velocity of propagation ξ of the shock wave greatly surpasses the flow velocities and the velocity of sound in the gas in its original condition. Equations (2) and (3) can then be simplified to:

$$u_1 = \frac{2\xi}{k+1} \quad ; \quad \varrho_1 = \frac{k+1}{k-1}\varrho_0 \quad ; \quad p_1 = \frac{k+1}{2}\varrho_0 u_1^2. \quad . \quad . \quad (10)$$

We take $k = 5/3$, which gives:

$$u_1 = \tfrac{3}{4}\xi \quad ; \quad \varrho_1 = 4\varrho_0 \quad ; \quad p_1 = \tfrac{4}{3}\varrho_0 u_1^2. \quad . \quad . \quad . \quad (10a)$$

Making use of (9) we further have:

$$dx/dt = \xi = 4\,S' = 4\,ds/dt \quad ; \quad u_1 = 3\,S' \quad . \quad . \quad . \quad (10b)$$

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[6]) The notation S' for the value of ds/dt along the path of the shock wave in the x, t-diagram was applied in the thesis, p. 13, and in the paper quoted in footnote 2), p. 268, eq. (13).

and consequently:

$$X = 4\,s \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (10c)$$

along the trace of the shock wave.

The Lagrangian equation of motion of the gas behind the shock wave is:

$$\frac{du}{dt} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} - g;$$

expressed with the aid of the function φ this becomes:

$$\frac{\partial^2 \varphi}{\partial t^2} = -\frac{1}{\varrho_1}\frac{\partial}{\partial s}\left(p_1\,\Phi^{-5/3}\right) - g. \quad . \; . \; . \; . \; . \; . \; . \qquad (11)$$

In the thesis solutions have been considered for which the function Φ was of the form:

$$\Phi = u\,(t)^3 / v\,(s)^3. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (12)$$

Working with equation (11) we assume the following expression for $\varphi(s;t)$:

$$\varphi = u\,(t)^3 \int^{x} \frac{ds}{v\,(s)^3} - a\,t - \tfrac{1}{2}\,g t^2 \,. \quad . \; . \; . \; . \; . \; . \qquad (13)$$

where a is a constant; the function Φ then again obtains the form (12). When (13) is substituted into (11), it is found that $u(t)$ must satisfy the equation:

$$u^5\,\frac{d^2 u^3}{dt^2} = \text{constant}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (14)$$

The solution of this equation has been given at p. 41 of the thesis and has the form:

$$\sqrt{E}\cdot t = (u^2 + \lambda^2)^{3/2} - 3\,\lambda^2\,(u^2 + \lambda^2)^{1/2} + \text{constant} \,. \quad . \; . \; . \qquad (15)$$

where E and λ are constants (in terms of the latter the constant of eq. (14) has the value $-\frac{1}{3}E\,\lambda^2$). We note:

$$\frac{du}{dt} = u' = \frac{\sqrt{E}}{3}\,\frac{(u^2 + \lambda^2)^{1/2}}{u^3}. \quad . \; . \; . \; . \; . \; . \; . \qquad (15a)$$

The origin for t will be defined in such a way that the constant on the right hand side of (15) is zero.

Since the trace of the shock wave in the x, t-plane is given by the relation $\Phi = 1$, or:

$$v\,(s) = u\,(t),$$

we obtain:

$$S' = \frac{ds}{dt} = \frac{u'}{v'} = \frac{\sqrt{E}}{3}\,\frac{(v^2 + \lambda^2)^{1/2}}{v'\,v^3}. \quad . \; . \; . \; . \; . \; . \; . \qquad (16)$$

Here we have replaced u by v in the expression for u', as the formula refers to conditions existing along the trace of the shock wave.

Now according to (10b) $u_1 = 3\,S'$; hence the condition to be fulfilled by $\varphi(s;t)$ becomes:

$$(\partial\varphi/\partial t)_{u=v} = 3\,S' \qquad \qquad (17)$$

4. We calculate $\partial\varphi/\partial t$ from (13) and again substitute v for u; the following relation is then found:

$$\left.\begin{aligned} \sqrt{E}\,\frac{(v^2+\lambda^2)^{1/2}}{v}\int\frac{ds}{v^3} - a - \frac{g}{\sqrt{E}}\left\{(v^2+\lambda^2)^{3/2} - 3\,\lambda^2\,(v^2+\lambda^2)^{1/2}\right\} = \\ = \sqrt{E}\,\frac{(v^2+\lambda^2)^{1/2}}{v'\,v^3} \end{aligned}\right\} \qquad (18)$$

This is an integro-differential equation for the function $v(s)$ which can be turned into an ordinary differential equation of the second order for v. It is a consequence of the special choice made for the function $\varphi(s;t)$ that the condition to be fulfilled at the trace of the shock wave gives sufficient information to fix the form of the factor $v(s)$. The differential equation for v, after a slight re-arrangement, can be brought into the form:

$$-\frac{v''}{v'^2\,v^3} - \frac{3}{v^4} = -\frac{a\,\lambda^2}{\sqrt{E}}\,\frac{v'}{v\,(v^2+\lambda^2)^{3/2}} - \frac{g}{E}\,v'\left(3\,v - \frac{2\,\lambda^2}{v}\right) \quad . \qquad (18a)$$

which can be integrated into:

$$\frac{1}{v'\,v^3} = \frac{a}{\lambda\sqrt{E}}\left\{\operatorname{arctgh}\frac{\lambda}{(v^2+\lambda^2)^{1/2}} - \frac{\lambda}{(v^2+\lambda^2)^{1/2}}\right\} + \beta - \frac{g}{E}\left(\tfrac{3}{2}\,v^2 - 2\,\lambda^2\ln v\right),$$

where β is a new constant. From this we obtain:

$$\left.\begin{aligned} \frac{ds}{dv} = \frac{1}{v'} = \frac{a\,v^3}{\lambda\sqrt{E}}\left\{\operatorname{arctgh}\frac{\lambda}{(v^2+\lambda^2)^{1/2}} - \frac{\lambda}{(v^2+\lambda^2)^{1/2}}\right\} + \\ + \beta\,v^3 - \frac{g}{E}\left(\tfrac{3}{2}\,v^5 - 2\,\lambda^2\,v^3\ln v\right) \end{aligned}\right\} \qquad (19)$$

A second integration gives:

$$\left.\begin{aligned} s = \frac{a\,v^4}{4\lambda\sqrt{E}}\left\{\operatorname{arctgh}\frac{\lambda}{(v^2+\lambda^2)^{1/2}} - \frac{(\lambda\,v^2 - 2\,\lambda^3)\,(v^2+\lambda^2)^{1/2}}{v^4}\right\} + \\ + \frac{\beta v^4}{4} - \frac{g}{E}\left(\frac{v^6}{4} - \frac{\lambda^2}{2}\,v^4\ln v + \frac{\lambda^2 v^4}{8}\right) \end{aligned}\right\} \qquad (20)$$

This equation fixes the relation between v and s. Since the relation between u and t is given by (15), the function Φ can now be considered as known. In order to obtain φ we have to carry out one further integration and arrive at:

$$\left.\begin{aligned} \varphi = u^3\int\frac{dv}{v'\,v^3} - a\,t - \tfrac{1}{2}\,gt^2 = u^3\left[\frac{a\,v}{\lambda\sqrt{E}}\operatorname{arctgh}\frac{\lambda}{(v^2+\lambda^2)^{1/2}} + \right. \\ \left. + \beta\,v - \frac{g}{E}\left(\frac{v^3}{2} - 2\,\lambda^2 v\ln v + 2\,\lambda^2\,v\right)\right] - a\,t - \tfrac{1}{2}\,gt^2. \end{aligned}\right\} \qquad (21)$$

The value of X follows from (10c); in order to have X as a function of t, it is necessary now to substitute u in the place of v. Differentiation with respect to t then gives:

$$\left.\begin{aligned}\xi = \frac{4\sqrt{E}}{3}\,(u^2+\lambda^2)^{1/2}\Bigg[\frac{a}{\lambda\sqrt{E}}\left\{\operatorname{arctgh}\frac{\lambda}{(u^2+\lambda^2)^{1/2}} - \frac{\lambda}{(u^2+\lambda^2)^{1/2}}\right\} + \\ + \beta - \frac{g}{E}\left(\tfrac{3}{2}\,u^2 - 2\lambda^2 \ln u\right)\Bigg]\end{aligned}\right\} \quad (22)$$

which also can be obtained from the relation $\xi = 4\,S' = 4\,u'/v'$.

5. In this way we have derived a complete expression for φ without making full use of the equation of motion (11). It will be evident that the latter equation can now be satisfied only provided a proper choice is made for the course of the density ϱ_0 as a function of the value of the parameter s along the trace of the shock wave.

The values of ϱ_1 and p_1 occurring in (11) are related to ϱ_0 by the expressions given in (10a). When u_1 is replaced by $3\,S'$ and use is made of (16), we find:

$$p_1 = \frac{4\,E}{3}\,\varrho_0\,\frac{v^2+\lambda^2}{v'^2\,v^6}.$$

This value must be introduced into (11); at the same time the expressions (13) for φ and (12) for Φ are substituted. The factor depending on t is eliminated by means of (14), inserting the value $-\frac{1}{3}\,E\,\lambda^2$ of the constant. The following equation is obtained in this way:

$$\lambda^2 \int \frac{ds}{v^3} = \frac{1}{\varrho_0}\,\frac{d}{ds}\left(\varrho_0\,\frac{v^2+\lambda^2}{v'^2\,v}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

We can eliminate $\int ds/v^3$ by making use of (18); there is some advantage in eliminating also the constant a by making use of (18a), which leads to:

$$\lambda^2 \int \frac{ds}{v^3} = \frac{v''\,(v^2+\lambda^2)}{v'^3\,v} + \frac{3\,v^2+4\,\lambda^2}{v'\,v^2} - \frac{3\,g}{E}\,v^5.$$

It is now possible to transform eq. (23) into:

$$\frac{d}{ds}\ln\left(\varrho_0\,\frac{v^2+\lambda^2}{v'^2\,v}\right) = \frac{v''}{v'} + \frac{4\,v'}{v} - \frac{v'\,v}{v^2+\lambda^2} - \frac{3\,g}{E}\,\frac{v'^2\,v^6}{v^2+\lambda^2},$$

which leads to the integral:

$$\ln\left(\varrho_0\,\frac{v^2+\lambda^2}{v'^2\,v}\right) = \ln\frac{v'\,v^4}{(v^2+\lambda^2)^{1/2}} - \frac{3\,g}{E}\int dv\,\frac{v'\,v^6}{v^2+\lambda^2}. \quad . \quad . \quad (24)$$

Although the remaining integral is complicated since v' must be obtained from (19), it will be seen that in principle the problem has been solved. With $g = 0$ the solution given by (24) is already complete.

We hope to bring a discussion of the results in a continuation of this paper; attention will then be given to the values to be chosen for the constants E, λ, α, β.

Without going into details a limiting case may be considered already here. We take $g = 0$.

From (15) and (15a) it will be seen that u increases with t and that ultimately $u^3 \backsim t$. In eq. (22) the important term for large u is:

$$\xi \cong \frac{4\beta\sqrt{E}}{3}\, u \backsim t^{1/3}.$$

Hence the velocity of the shock wave front increases with time.

With $g = 0$ equation (24) gives:

$$\varrho_0 = \text{const.}\ v'^3 v^5 (v^2 + \lambda^2)^{-3/2}.$$

It must be remembered that along the trace of the shock wave we have $v = u$. For large values of u and v eq. (19) gives:

$$1/v' \cong \beta v^3,$$

so that we obtain:

$$\varrho_0 \cong \text{const.}\ v^{-7}.$$

At the same time eq. (20) reduces to: $s \cong \frac{1}{4}\beta v^4$, from which it follows that:

$$X \cong \beta v^4$$

along the trace of the shock wave.

Hence we arrive at the result that ultimately:

$$\varrho_0 \cong X^{-7/4}$$

(this result is valid for all values of t when $\lambda = 0$, which brings us back to a case already considered in the thesis of the first author, p. 22).

A density distribution in the original gas, proportional to $x^{-7/4}$, in the absence of motion, can be maintained only by a proper gravitational field. Since we had assumed that the velocity ξ of the shock wave is much larger than the velocity of sound in the original gas, we are at liberty to suppose that the temperature of this gas is very low. Then, however, a very small intensity of gravity is sufficient to produce a certain density gradient. Hence we may suppose that g is very small and that its influence on the shockwave phenomena will be restricted to a small correction.

We hope to come back to some numerical data.

Potchefstroom (Transvaal) and *Delft*, June–October 1949.

(To be continued.)

Zoology. — *Sacculina cuspidata nov. spec., with notes on variation in Loxothylacus carinatus (Kossm.).* By H. BOSCHMA.

(Communicated at the meeting of October 29, 1949.)

The new species of Rhizocephalan parasite described in the present paper bears some resemblance to *Loxothylacus carinatus* (Kossm.). In the two species not only the excrescences of the external cuticle are of a similar shape, but one of the specimens of *Sacculina cuspidata* in its male organs presents a structure which is strongly similar to that prevailing in *Loxothylacus.* As, however, its other characters are those typical of the genus *Sacculina,* and as the second specimen in all its characters proves to belong to this genus, the new species is placed in the genus *Sacculina.* A comparison of the male organs and especially of the excrescences of the external cuticle of *Loxothylacus carinatus* with the corresponding structures in *Sacculina cuspidata* proves that the two forms at least specifically are distinctly different. The placing of the new species in the genus *Sacculina,* however, for the present remains open to some doubt.

Besides details of the external and internal structure of the new species in the present paper some data are given concerning variation in *Loxothylacus carinatus.*

Sacculina cuspidata nov. spec.

Material examined:

Snellius Expedition, Tidore, September 24—29, 1929, 1 specimen (holotype), 6 × 4 × 2½ mm, on *Pseudozius caystrus* (Adams & White).

Snellius Expedition, Mamudju (Celebes), August 4—5, 1929, shore or reef, 1 specimen, 4½ × 3 × 1½ mm, on *Chlorodopsis* spec.

Specific characters. Mesentery complete. Visceral mass attached to the region of the stalk. Male genital organs in the visceral mass, the two testes of approximately equal size, straight or slightly bent in an anterior direction. Colleteric glands with a comparatively small number of branched canals (25—30 in a longitudinal section of the most strongly branched region). External cuticle of the mantle with excrescences consisting of a kind of chitin differing from that of the main layers. The excrescences have a more or less cylindrical body, a slender root penetrating into the cuticle, and at their extremities bear a number of sharp pointed spines. The length of the excrescences is from 30 to 73 μ, that of the part extending above the cuticle 18 to 60 μ, the spines have a length of 9 to 27 μ. Internal cuticle of the mantle with numerous retinacula, each with 1 to 3 spindles; the latter have a length of 13 to 18 μ.

The two specimens, though living on different hosts, are of a strikingly similar shape, being more or less trapezoid (fig. 1*a*, *b*). The narrow mantle opening, in the anterior part of the left side, does not protrude above its surroundings. In the greater part of the mantle the surface is comparatively smooth, with the exception of the anterior region in which in both specimens there are a number of more or less pronounced grooves running in a dorso-ventral direction.

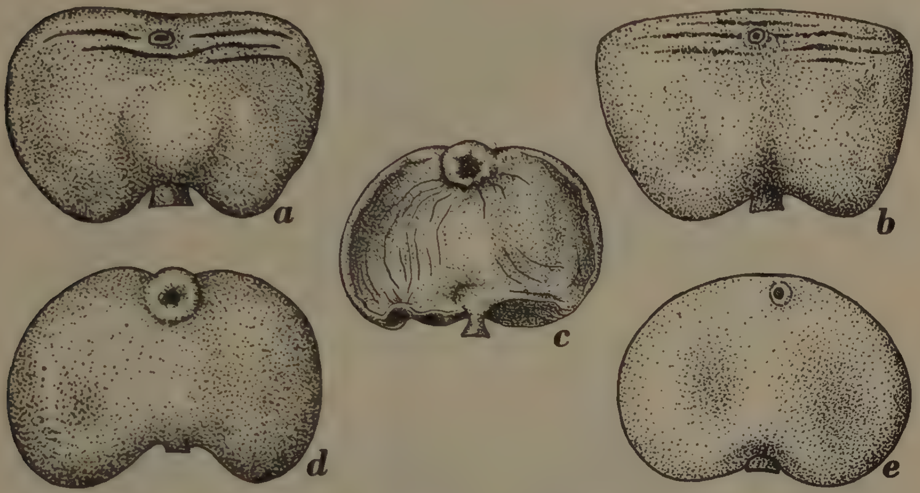

Fig. 1. *Sacculina cuspidata*; *a*, specimen on *Chlorodopsis* spec.; *b*, specimen on *Pseudozius caystrus*. *Loxothylacus carinatus*; *c*, specimen from unknown host; *d*, specimen on unidentified Portunid crab; *e*, specimen on *Thalamita admeta*. All specimens left side. *a*, × 9; *b*, × 6½; *c*, × 5; *d*, × 4; *e*, × 5½.

In the holotype the visceral mass is attached to the region of the stalk (fig. 2 *a*). Here the vasa deferentia are found which in their ventral parts are narrow canals that may show a few ridges on their internal walls. Gradually, towards a more dorsal region, the male organs enlarge into rather voluminous glands of slightly flattened shape so that their greater diameter is more or less in an antero-posterior direction (fig. 2 *b*—*e*). At first their wall is rather thick, leaving inside a narrow, somewhat twisted cavity only; towards the dorsal extremity at least the anterior part of the testes obtains a much wider cavity. The testes are not curved to any extent, but the irregular cavity gives the impression as if the testes really had a distinct curve. The section of, e.g., fig. 2 *d* at first sight is so strikingly similar to corresponding sections of specimens undoubtedly belonging to *Loxothylacus* that it seems wrong to place the parasite in the genus *Sacculina*. When, however, curved testes occur (as they are typical for the genera *Loxothylacus, Heterosaccus* and *Drepanorchis*) the concave side of the curve is turned towards the ventral half of the body. If really the curved cavity of the testes in the present specimen is an indication of curved testes the curvature is turning its concave side towards the dorsal half of the body. It seems, therefore, safe to assume that the double cavity in sections as that of fig. 2 *d* is a result of the irregularly twisted cavity of the testes on account of the strong and uneven development of the ridges on the inner walls. The two male organs are of about equal size, they are lying close together but remain completely separated.

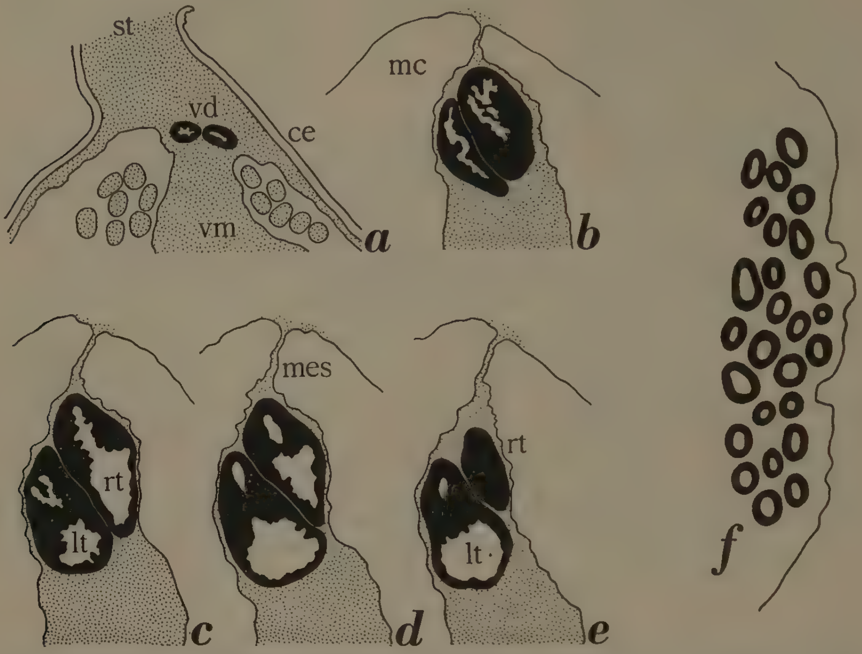

Fig. 2. *Sacculina cuspidata*, specimen on *Pseudozius caystrus*. *a*—*e*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *f*, longitudinal section of one of the colleteric glands. *ce*, external cuticle; *lt*, left testis; *mc*, mantle cavity; *mes*, mesentery; *rt*, right testis; *st*, stalk; *vd*, vasa deferentia; *vm*, visceral mass. *a*—*e*, × 30; *f*, × 107.

The colleteric glands are found near the anterior extremity of the visceral mass. They contain a more or less flattened, rather compact mass of branched tubes. In a section of the most strongly divided region of one of these glands (fig. 2 *f*) 26 canals could be counted.

The external cuticle of the mantle is densely covered with excrescences of a kind of chitin (more hyaline and not as easily stained) different from that of the main layers of this cuticle. The excrescences consist of more or less cylindrical, usually somewhat contorted bodies which with slender roots are fastened in sockets of the main layers (fig. 3 *a*—*c*). On the free extremities of the cylindrical bodies there is a row of sharp spines implanted along the margin. The figures show the variation in size and in shape of the excrescences, the measurements of the component parts are given above (cf. specific characters, in which also the details of the excrescences of the other specimen have been taken into account). In surface view the excrescences appear as those of the other specimen (cf. fig. 5 *a*).

The internal cuticle of the mantle shows numerous retinacula which are more or less regularly distributed over its surface. These consist of a slender basal part and one to three spindles; the spindles, which possess small barbs, vary in length from 13 to 18 μ (fig. 3 *d*).

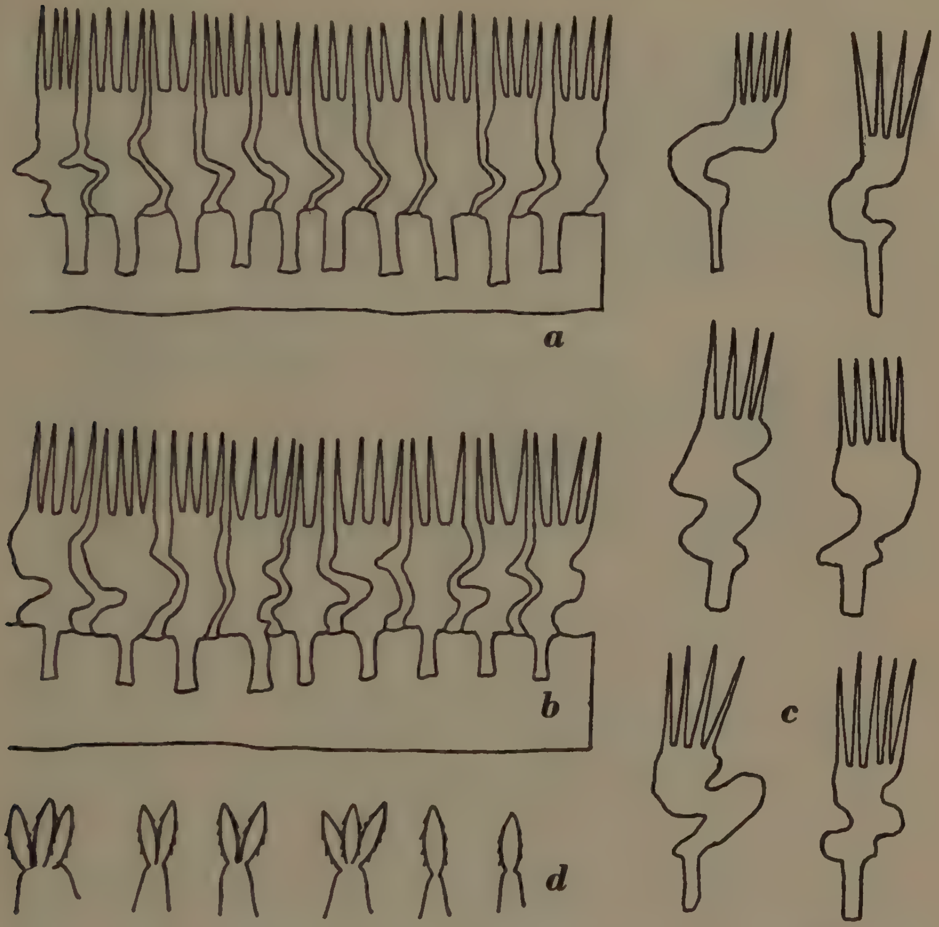

Fig. 3. *Sacculina cuspidata*, specimen on *Pseudozius caystrus*. *a, b,* sections of the external cuticle; *c*, isolated excrescences; *d*, retinacula. × 530.

In the specimen on *Chlorodopsis* spec. again the visceral mass is attached to the region of the stalk (fig. 4 *a*). Here again the vasa deferentia in their ventral part are narrow canals, which in a more dorsal region gradually become wider, whilst at first their cavity remains narrow as a result of ridges on their inner walls (fig. 4 *b*). In a still more dorsal region the testes gradually increase in size whilst in connexion with a varying thickness of the wall the cavities correspondingly vary in size (fig. 4 *c*—*e*). In this specimen again the two male organs are of equal size. They do not show a tendency for a curve towards the anterior region though they are slightly flattened in a lateral direction.

The colleteric glands are found in the neighbourhood of the anterior extremity of the visceral mass, they are more or less flattened and have a rather compact mass of branched canals. In a longitudinal section of the most strongly divided region of one of the colleteric glands there are 29 canals (fig. 4 *f*).

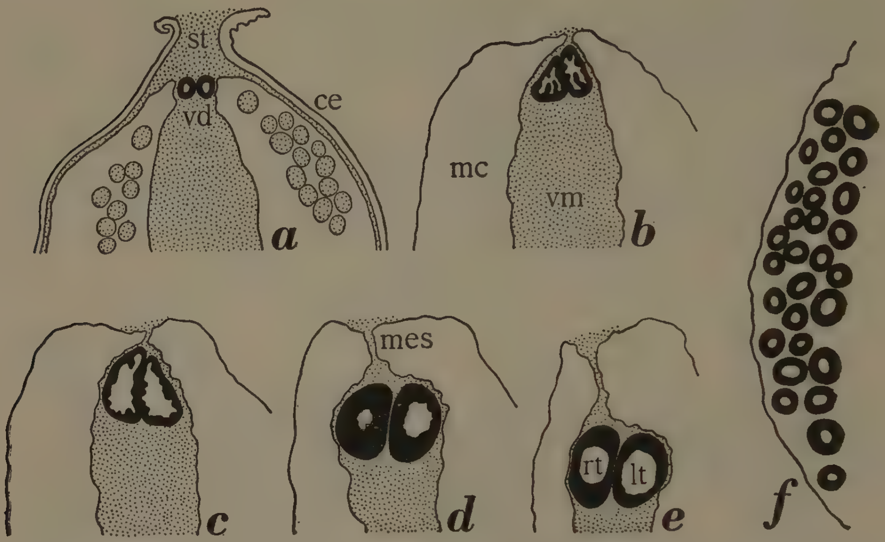

Fig. 4. *Sacculina cuspidata*, specimen on *Chlorodopsis* spec. *a*—*e*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *f*, longitudinal section of one of the colleteric glands; *ce*, external cuticle; *lt*, left testis; *mc*, mantle cavity; *mes*, mesentery; *rt*, right testis; *st*, stalk; *vd*, vasa deferentia; *vm*, visceral mass. *a*—*e*, × 30; *f*, × 127.

The excrescences of the external cuticle of the mantle in every respect correspond with those of the type specimen. A certain amount of variation in shape and size is to be observed (fig. 5 *b*—*e*), in the excrescences as a whole as well as in their component parts, but in general they are not different from those of the other specimen. In surface view (fig. 5 *a*) they appear to be rather crowdedly covering the surface of the cuticle.

No retinacula were found on the internal cuticle, though these structures undoubtedly are present here. It often depends upon the part of the mantle examined or on the state of preservation of the specimen whether the retinacula are easily to be found or seem to be missing.

The specimen on *Chlorodopsis* spec. has all the characters peculiar to the genus *Sacculina*. It has testes with a straight course and, moreover, the visceral mass is attached to the region of the stalk. This latter character is also present in the specimen on *Pseudozius caystrus*, but here, as remarked above, the testes differ from those normally found in *Sacculina*. It seems safe, however, to keep the species in the latter genus.

The excrescences of the external cuticle of *Sacculina cuspidata* are strongly similar to those of *Loxothylacus carinatus*. The chief difference is that of the spines which are much larger and stronger developed than those in the latter species. Moreover, in *L. carinatus* the retinacula have a greater number of spindles.

Another species with similar excrescences is *Loxothylacus ihlei* (cf. BOSCHMA, 1949). Here again the spines of the excrescences are much smaller than those of *S. cuspidata,* whilst the retinacula in the two species are entirely different, those of *L. ihlei* having numerous spindles of very large size and those of *S. cuspidata* having a few spindles of moderate size only.

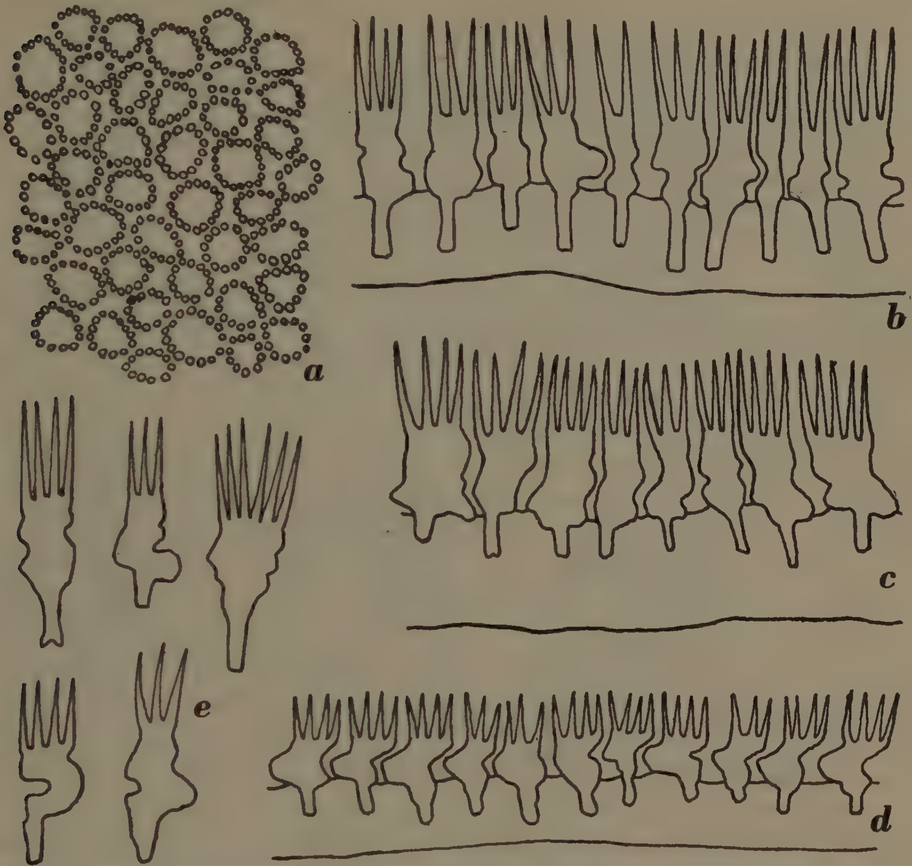

Fig. 5. *Sacculina cuspidata,* specimen on *Chlorodopsis* spec. *a,* excrescences in surface view (basal parts of spines drawn only); *b—d,* sections of various parts of the external cuticle; *e,* isolated spines. × 530.

Loxothylacus carinatus (Kossm.).

Sacculina carinata Kossmann, 1872, 1874; Van Kampen & Boschma, 1925.
Drepanorchis carinata Boschma, 1928.
Loxothylacus carinatus Boschma, 1931 *a,* 1931 *b.*

Material examined:

Siboga Expedition, Sta. 220, Binongko, Tukang Besi Islands, reef, 1 specimen, 8½ × 4½ × 2½ mm, on *Caphyra levis* (A. Milne Edwards).

Banda, SEMMELINK leg., 1881, 1 specimen, 9½ × 5½ × 4 mm, on *Thalamita prymna* (Herbst) (collection Museum Leiden).

East Asia, Capt. SUENSON leg., Miss ANTHON don., 19. IV. 1911, 1 specimen, 8 × 5 × 2½ mm, on *Neptunus* (*Amphitrite*) *hastatoides* (Fabr.) (collection Museum Copenhagen).

Dr. TH. MORTENSEN's Pacific Expedition, off Jolo (Sulu Islands), 33—35 m, dredge, 19. III. 1914, 1 specimen, 5¼ × 4 × 2½ mm, on *Lissocarcinus polybioides* Adams & White (collection Museum Copenhagen).

Hong Kong, Putnam leg., 1 specimen, 7 × 5 × 1½ mm, detached from unknown host (collection Museum of Comparative Zoölogy, Cambridge, Mass.).

Snellius Expedition, Obi latu, April 23—27, 1930, shore or reef, 1 specimen, 10 × 7 × 3 mm, on an unidentified Portunid crab.

Snellius Expedition, Amboina, October 17, 1930, 1 specimen, 7 × 5 × 2½ mm, on *Thalamita admeta* (Herbst).

The external shape of the known specimens of *Loxothylacus carinatus* is more or less similar. As a rule the specimens are oval or somewhat panduriform, the larger diameter being in the median region of the body half-way between the anterior and the posterior regions. This shape is that of the type specimen (KOSSMANN, 1872, Pl. I fig. 12; 1874, Pl. V fig. 12); here the mantle opening is rather small whilst its surroundings do not protrude noticeably above the surface. The specimens on *Caphyra levis* (VAN KAMPEN & BOSCHMA, 1925, Pl. I fig. 12) and on *Thalamita prymna* are panduriform whilst the mantle opening is surrounded by a comparatively thick wall protruding over the neighbourhood. The specimen on *Neptunus hastatoides* (BOSCHMA, 1931 *a*, fig. 33 *m*) has a similar shape and a somewhat protruding mantle opening. In the specimen on *Lissocarcinus polybioides* (loc. cit., fig. 33 *k*) the larger diameter is in the posterior half, the mantle opening is raised above the surface, and in the posterior region there is a pronounced keel (as in the type specimen). The specimen from unknown host (fig. 1 *c*) is rather flat, as the mantle cavity does not contain eggs. Here the larger diameter is in the posterior half, the surroundings of the mantle opening protrude distinctly above the surface, and the mantle shows a number of grooves and wrinkles. The specimen on the unidentified Portunid crab (fig. 1 *d*) again is more or less panduriform, its mantle opening protrudes distinctly above the surface of the mantle. The specimen on *Thalamita admeta* (fig. 1 *e*) is irregularly oval; here the mantle opening is not raised above the surface of the mantle.

The excrescences of the external and the internal cuticle of the specimens on *Caphyra levis, Thalamita prymna, Neptunus hastatoides* and *Lissocarcinus polybioides* have been described and figured in previous papers (VAN KAMPEN & BOSCHMA, 1925; BOSCHMA, 1931 *a*, 1931 *b*). In these papers also some details are given of the male organs and the colleteric glands of three of the specimens; the sections made from the specimen on *Caphyra levis* were a complete failure so that no details of the internal organs could be noted. A more complete description of the structure of the male organs and of the colleteric glands of the three other specimens is given here to draw attention to the rather pronounced individual variation of the specimens.

In the specimen on *Thalamita prymna* the visceral mass is attached to the region of the stalk (fig. 6 *a*, in this section by means of the mesentery),

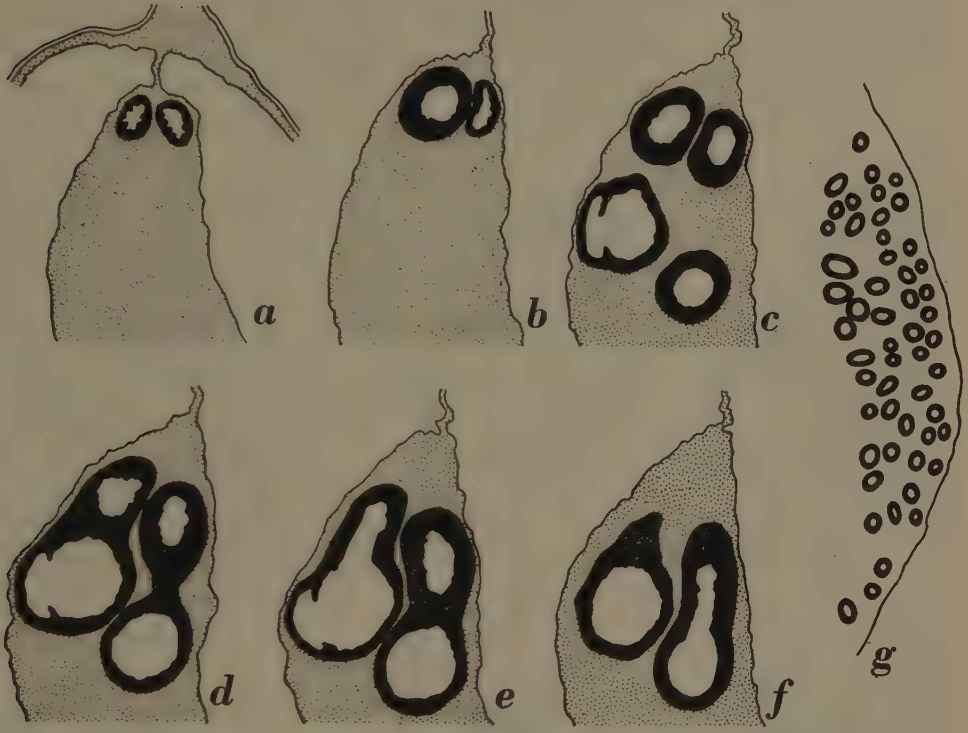

Fig. 6. *Loxothylacus carinatus,* specimen on *Thalamita prymna.* *a—f*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *g*, longitudinal section of one of the colleteric glands. *a—f*, × 14½; *g*, × 64.

a feature uncommon in *Loxothylacus*. Here the vasa deferentia are narrow canals with a few ridges on their inner walls. Towards a more dorsal region the male organs become wider (fig. 6 *b*) and soon in the sections the anterior, terminal part of the testes appears (fig. 6 *c*). Sections from a still more dorsal region show various parts of the curvature of the testes (fig. 6 *d—f*). In this specimen the two male organs are of similar shape and size, they are rather strongly curved so that the anterior part remains beside the posterior part. The cavities of the testes are comparatively wide.

The colleteric glands are found in the central parts of the lateral surfaces of the visceral mass. They are more or less flattened and contain a fairly large number of canals. In the figured section, from the most strongly divided region (fig. 6 *g*), there are 53 canals.

In the specimen on *Neptunus hastatoides* the visceral mass is attached to the mantle at some distance from the region from which the stalk takes its origin (fig. 7 *b*). Here again the male organs in their ventral part are narrow canals (fig. 7 *b*). Towards a more dorsal region they increase in size (fig. 7 *c*), and the ridges developing on their inner walls may become more strongly pronounced (fig. 7 *d*). From here onwards the two male organs are of an entirely different shape and attain a totally different size. The left testis in its dorsal part diminishes in size and develops into a rather narrow canal running in an anterior direction (fig. 7 *e*, *f*). The right testis enormously increases in size, it extends for some distance in an anterior direction (fig. 7 *e*, *f*), whilst its terminal part is distinctly curved towards the ventral surface (fig. 7 *d*). The cavity of the right testis is very wide, that of the left testis is extremely narrow; apparently only the right testis was functional. The male organs extend somewhat farther towards the anterior region than in the specimen on *Thalamita prymna*.

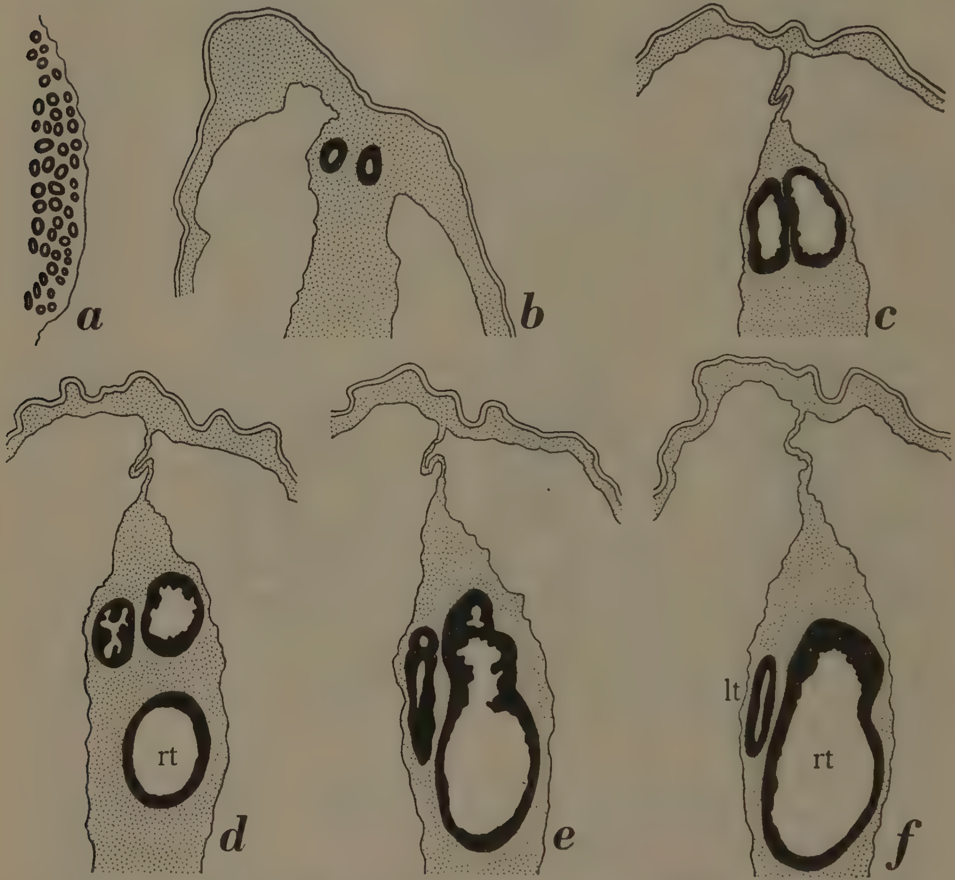

Fig. 7. *Loxothylacus carinatus*, specimen on *Neptunus hastatoides*. *a*, longitudinal section of one of the colleteric glands; *b*—*f*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding. *lt*, left testis; *rt*, right testis. *a*, × 53; *b*—*f*, × 30.

The colleteric glands occupy a central position in the lateral surfaces of the visceral mass. They are more or less flattened and contain a rather compact mass of tubes. As in the specimen described above the canals do not possess chitin, so that the glandular epithelium is drawn. A section of the most strongly divided region of one of the colleteric glands (fig. 7 *a*) shows 48 canals.

The visceral mass of the specimen on *Lissocarcinus polybioides* is attached to the mantle next to the stalk, so that the specimen shows this character peculiar to the genus *Loxothylacus* (fig. 8 *a*). In this specimen the vasa deferentia remain narrow for a long distance (fig. 8 *a—e*), in their most dorsal part they are rather contorted (fig. 8 *e*). After the passing of the vasa deferentia into the testes the latter enlarge into comparatively wide sacs, the terminal part of which is distinctly protruding in a ventral direction. The male organs, therefore, show a distinct curve. The two testes are of approximately equal size, but the one is placed behind the other, so that in the most dorsal region (fig. 8 *f*) only the left testis, and in the most ventral region (fig. 8 *a, b*) only the right testis is to be seen in the sections. Here again the curvature of the male organs is very distinct.

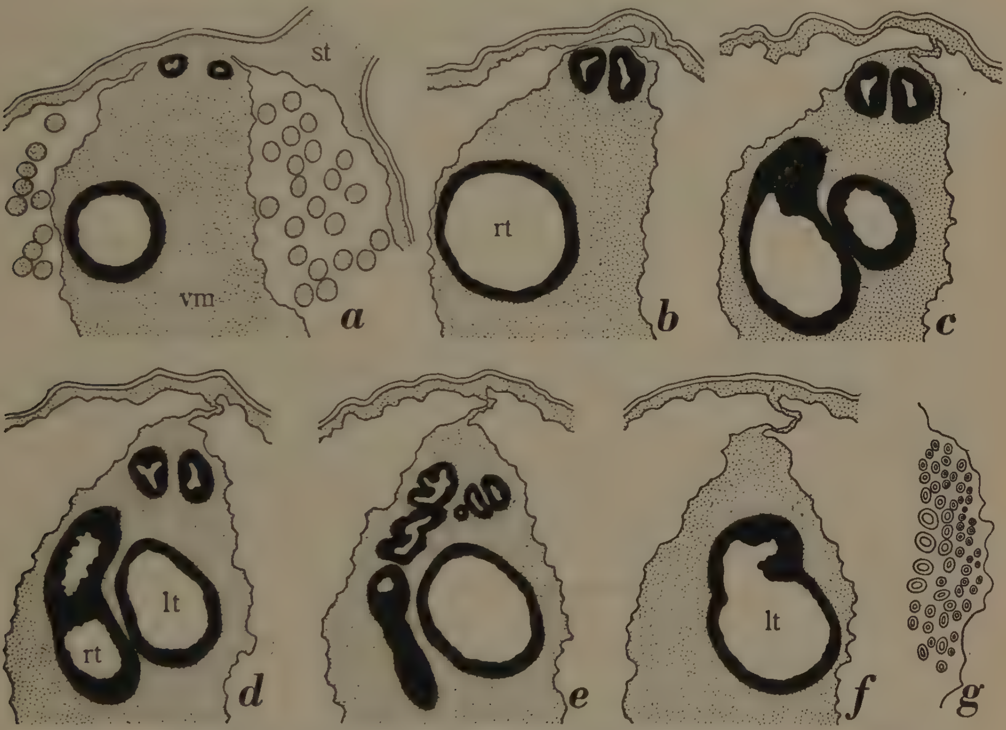

Fig. 8. *Loxothylacus carinatus*, specimen on *Lissocarcinus polybioides*. *a—f*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *g*, longitudinal section of one of the colleteric glands. *lt*, left testis; *rt*, right testis; *st*, stalk; *vm*, visceral mass. *a—f*, × 30; *g*, × 53.

The colleteric glands are found in the central part of the lateral surfaces of the visceral mass. They consist of a more or less flattened mass of branched tubes, which here possess chitinous contents. In the most strongly divided region of these glands a longitudinal section (fig. 8 *g*) shows 60 canals.

The peculiarities of the three other specimens listed above will be described in a later paper. A few remarks are given here.

In the specimen from unknown host the right testis has a large size, the left is rudimentary. The right testis is distinctly curved. The colleteric glands contain 35 canals in a section of the most strongly divided part.

In the specimen on the unidentified Portunid crab the male organs are of approximately equal size, both are distinctly curved. A section of the most strongly divided region of the colleteric glands contains 58 canals.

In the specimen on *Thalamita admeta* the right testis is of large size, the left testis is much smaller. Both male organs show a distinct curve. A section of the most strongly divided part of the colleteric glands has 36 canals.

Consequently in three of the specimens the testes are of approximately equal size, whilst in the three others the left testis is decidedly smaller than the right. As in other respects the specimens form a homogeneous group this difference in size of the male organs cannot be regarded as a character of specific value.

It is interesting that the hosts of *Sacculina cuspidata* both belong to the family Xanthidae, whilst all the hosts of *Loxothylacus carinatus* (as far as these are known) are crabs of the family Portunidae. The type specimen (KOSSMANN, 1872, p. 32; 1874, p. 128) was a parasite of "*Lupea* sp. aff. *L. hastatae*" (obviously a species of the genus *Neptunus*), belonging to the same family. The close relation of the different hosts gives additional evidence for the opinion that KOSSMANN's *Sacculina carinata*, the anatomy of which is imperfectly known, belongs to the same species as the specimens dealt with above.

Loxothylacus ihlei, a species with excrescences of the external cuticle of a similar shape and size as those of *L. carinatus*, is a parasite of the Portunid crab *Scylla serrata* (Forsk.) (cf. BOSCHMA, 1949). Here, however, the retinacula are of much larger size than those of *L. carinatus*, the difference in size and shape between the two testes is much more pronounced than in the specimens dealt with above that show this difference, and the colleteric glands have a far larger number of canals (up to 168 in a longitudinal section).

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Mathematics. — *Affine embedding theory III: Integrability conditions.* By V. HLAVATÝ. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of September 24, 1949.)

Synopsis. This paper is a continuation of two previous papers *Affine embedding theory I* (p. 503) and *Affine embedding theory II* (p. 714), which will be referred to as *AI* and *AII*. First, condensed indices are introduced which enable us to simplify the FRENET formulae *AII* (2. 2) for an A_m in A_n. Then the integrability conditions of these formulas are established and their consequences for the embedding theory as well as for the external contact invariants are investigated.

1. *Condensed indices.*

1) Let us introduce the indices $A_q, B_q, C_q, D_q, \ldots, q = 2, \ldots, N$, which take the values from $m^{q-1}+1$ to $m^{q-1}+m^q$. Hence $A_q, B_q \ldots$ take m^q values, which is the number of values taken by any group of small latin indices, say $a_q\, a_{q-1} \ldots a_1$ $(a_q, a_{q-1}, \ldots, a_1 = 1, \ldots, m)$. Hence a one-to-one mapping of these two kinds of indices may be established. This will be achieved by means of the generalized KRONECKER deltas $\delta^{A_q}_{a_q \ldots a_1} = \delta^{a_q \ldots a_1}_{A}$ 1). The indices $A_q, B_q, \ldots$ will be referred to as *condensed indices.* By means of the generalized KRONECKER delta's we can build up *condensed* vectors X_{A_q}, Y^{A_q} starting with tensors $X_{a_q \ldots a_1}$, $Y^{a_q \ldots a_1}$

$$\left.\begin{aligned} X_{A_q} &= \delta^{a_q \ldots a_1}_{A_q}\, X_{a_q \ldots a_1} \quad , \quad X_{a_q \ldots a_1} = \delta^{A_q}_{a_q \ldots a_1}\, X_{A_q} \\ Y^{A_q} &= \delta^{A_q}_{a_q \ldots a_1}\, Y^{a_q \ldots a_1} \quad , \quad Y^{a_q \ldots a_1} = \delta^{a_q \ldots a_1}_{A_q}\, Y^{A_q}. \end{aligned}\right\} \quad . \quad . \quad (1,1)$$

The right hand equations are immediate consequences of the left hand equations as well as of the first one of the following equations

$$\delta^{A_q}_{a_q \ldots a_1}\, \delta^{b_q \ldots b_1}_{A_q} = \delta^{b_q \ldots b_1}_{a_q \ldots a_1}\,{}^{2)} \quad , \quad \delta^{A_q}_{a_q \ldots a_1}\, \delta^{a_q \ldots a_1}_{B_q} = \delta^{A_q}_{B_q}. \quad . \quad (1,2)$$

We do not need to emphasize the obvious relation

$$X_{A_q}\, Y^{A_q} = X_{a_q \ldots a_1}\, Y^{a_q \ldots a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1,3)$$

---

1) Thus for instance if $m = 2$, $q = 2$ we may put

$$\delta^{11}_3 = \delta^{12}_4 = \delta^{21}_5 = \delta^{22}_6 = 1, \text{ the remaining } \delta^{a_2 a_1}_{A_2} = 0.$$

2) $$\delta^{b_q \ldots b_1}_{a_q \ldots a_1} \equiv \delta^{b_q}_{a_q} \ldots \delta^{b_1}_{a_1}.$$

which follows from (1, 1) and (1, 2). An η-tensor can have two kinds of indices, as, for instance, $\delta^{A_q}_{a_q \dots a_1}$. Such a tensor will be referred to as a *mixed η-tensor*. In a similar manner a *mixed $(\xi\eta)$-tensor* may be defined.

2) If $X^{B_q \dots \nu \dots}_{A_q \dots a \dots}$ is a mixed $(\xi\eta)$-tensor its covariant derivative will be denoted by $D_c X^{B_q \dots \nu \dots}_{A_q \dots a \dots}$

$$\left.\begin{aligned} D_c X^{B_q \dots \nu \dots}_{A_q \dots a \dots} &= \nabla_c X^{B_q \dots \nu \dots}_{A_q \dots a \dots} - \Gamma_{A_q c}{}^{C_q} X^{B_q \dots \nu \dots}_{C_q \dots a \dots} \\ &\quad + \Gamma_{C_q c}{}^{B_q} X^{C_q \dots \nu \dots}_{A_q \dots a \dots} + \dots - \Gamma_{ac}{}^{b} X^{B_q \dots \nu \dots}_{A_q \dots b \dots} \end{aligned}\right\} . \qquad (1, 4)$$

where $\Gamma_{C_q e}{}^{D_q}$ are to be defined.

Theorem (1, 1). *A necessary and sufficient condition for*

$$a) \quad D_b \delta^{A_q}_{a_q \dots a_1} = 0 \quad \text{and} \quad b) \quad D_b \delta^{a_q \dots a_1}_{A_q} = 0 \; . \quad . \quad . \quad (1, 5$$

is

$$a \quad \Gamma_{B_q b}{}^{A_q} = \sum_{1}^{q} {}_r \, \Gamma_{a_r b}{}^{c_r} \delta^{c_q \dots c_{r+1} a_r c_{r-1} \dots c_1}_{B_q} \delta^{A_q}_{c_q \dots c_1} \; . \quad . \quad (1, 6a)$$

and this equation is equivalent to

$$\Gamma_{B_q b}{}^{A_q} = \sum_{1}^{q} {}_r \, \Gamma_{a_r b}{}^{c_r} \delta^{A_q}_{a_q \dots a_{r+1} c_r a_{r-1} \dots a_1} \delta^{a_q \dots a_1}_{B_q} \; . \quad . \quad . \quad (1, 6b)$$

Proof. We have $\partial_b \delta^{A_q}_{a_q \dots a_1} = 0$ and consequently

$$D_b \delta^{A_q}_{v_q \dots a_1} = \Gamma_{B_q b}{}^{A_q} \delta^{B_q}_{a_q \dots a_1} - \sum_{1}^{q} {}_r \, \Gamma_{a_r b}{}^{c_r} \delta^{A_q}_{a_q \dots a_{r+1} c_r a_{r-1} \dots a_1} . \quad (1,7)$$

Hence a necessary and sufficient condition for (1, 5*a*) reduces to (1, 6*b*) and this equation is obviously equivalent to (1, 6*a*). From (1, 2) and (1, 5*a*) we get

$$\delta^{A_q}_{a_q \dots a_1} D_b \delta^{a_q \dots a_1}_{B_q} = 0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 8)$$

and this equation is equivalent to (1, 5*b*).

Note. Throughout this paper we shall suppose the validity of (1, 6).

2. *Condensed* FRENET *formulae and their integrability conditions.*

Let us introduce the following condensed mixed $(\xi\eta)$-tensors related to the $(\xi\eta)$-tensors $H^{\nu}_{b_q \dots b_1}$ (cf. theorem (5, 2) in *AI*)

$$H^{\nu}_{B_q} = \delta^{b_q \dots b_1}_{B_q} H^{\nu}_{b_q \dots b_1} \; , \quad H^{\nu}_{b_q \dots b_1} = \delta^{B_q}_{b_q \dots b_1} H^{\nu}_{B_q} \; . \quad . \quad (2, 1)$$

$$(q = 2, \dots, N).$$

Because $H^{\nu}_{b_q \ldots b_2 b_1} = H^{\nu}_{b_q \ldots b_1 b_2}$, the ξ-vectors $H^{\nu}_{B_q}$ are *linearly dependent* even in the maximal case. — Moreover let us introduce the mixed η-tensors

$$\left.\begin{array}{l}
K_{a A_q}{}^{b} = K_{a a_q \ldots a_1}{}^{b} \, \delta^{a_q \ldots a_1}_{A_q} , \\
\qquad\qquad K_{a A_q}{}^{B_{q-r}} = \left\{ K_{a a_q \ldots a_{q-r}}{}^{b_{q-r}} \, \delta^{b_{q-r-1} \ldots b_1}_{a_{q-r-1} \ldots a_1} \right\} \delta^{a_q \ldots a_1}_{A_q} \, \delta^{B_{q-r}}_{b_{q-r} \ldots b_1} \\
L_{a A_N}{}^{B_q} = L_{a a_N \ldots a_q \ldots a_1}{}^{b_q \ldots b_1} \, \delta^{a_N \ldots a_1}_{A_N} \, \delta^{B_q}_{b_q \ldots b_1} , \quad \delta_{a A_u}{}^{B_{u+1}} = \delta^{B_{u+1}}_{a a_u \ldots a_1} \, \delta^{a_u \ldots a_1}_{A_u} , \\
(q = 2, \ldots, N \ , \ r = 1, 2, \ldots, q-2 \ , \ q - r \geqq 2 \ , \ u = 2, \ldots, N-1)
\end{array}\right\} \quad (2, 2)$$

which occur in the following

Theorem (2, 1). *The* FRENET *formula AII* (2, 2) *may be written in a condensed form*

$$\left.\begin{array}{l}
\text{a)} \quad D_b T^{\nu}_a = H^{\nu}_{ba} \quad , \quad \text{b)} \quad D_b H^{\nu}_{A_2} = \delta_{b A_2}{}^{A_3} H^{\nu}_{A_3} + K_{b A_2}{}^{c} T^{\nu}_c \\
\text{c)} \quad D_b H^{\nu}_{A_q} = \delta_{b A_q}{}^{A_{q+1}} H^{\nu}_{A_{q+1}} + \sum_{1}^{q-1}{}_{r} K_{b A_q}{}^{C_r} H^{\nu}_{C_r} \;^{3)} \\
\text{d)} \quad D_b H^{\nu}_{A_N} = L_{b A_N}{}^{B_N} H^{\nu}_{B_N} + \sum_{2}^{N-1}{}_{r} \left(K_{b A_N}{}^{C_r} + L_{b A_N}{}^{C_r} \right) H^{\nu}_{C_r} + K_{b A_N}{}^{c} T_c \\
\qquad\qquad \left(q = 3, \ldots, N-1 \quad , \quad H^{\nu}_{C_1} = T^{\nu}_{C_1} \right)^{3)}
\end{array}\right\} \quad (2, 3)$$

Proof. By virtue of (1, 5) we have

$$\left(D_b H^{\nu}_{a_q \ldots a_1} \right) \delta^{a_q \ldots a_1}_{A_q} = D_b H^{\nu}_{a_q \ldots a_1} \delta^{a_q \ldots a_1}_{A_q} = D_b H^{\nu}_{A_q} . \quad (2, 4)$$

On the other hand

$$\delta^{a_q \ldots a_1}_{A_q} H^{\nu}_{b a_q \ldots a_1} = \delta^{a_q \ldots a_1}_{A_q} \delta^{B_{q+1}}_{b a_q \ldots a_1} H^{\nu}_{B_{q+1}} = \delta_{b A_q}{}^{B_{q+1}} H^{\nu}_{B_{q+1}} . \quad (2, 5)$$

The proof follows at once from (2, 4), (2, 5) and *AII* (22).

In order to establish the integrability condition of (2, 3) (for $m > 1$) we first complete the sets of mixed η-tensors K and L by putting

$$\left.\begin{array}{l}
K_{a A_N}{}^{B_N} = K_{a A_x}{}^{B_N} = L_{a A_x}{}^{B_q} = L_{a A_N}{}^{b} = 0 \\
(x = 1, \ldots, N-1, \ q = 1, 2, \ldots, N)
\end{array}\right\} \quad \ldots \ldots \quad (2, 6)$$

---

[3] If $N = 2$, then we have only (2, 3*a*) and (2, 3*d*), where $\sum_{2}^{N-1} \equiv 0$. Moreover we assume here and later on that small latin indices which take the values from 1 to m may be denoted also by $A_1, B_1, C_1, \ldots$.

and introduce the following abbreviations

$$\left.\begin{array}{l} W_{b\,A_x}^{\;\;B_s}=\delta_{b\,A_x}^{\;\;B_s}\,,\; W_{b\,A_x}^{\;\;B_y}=K_{b\,A_x}^{\;\;B_y}+L_{b\,A_x}^{\;\;B_y}\,,\; W_{b\,A_N}^{\;\;B_z}=K_{b\,A_N}^{\;\;B_z}+L_{b\,A_N}^{\;\;B_z} \\ (x=1,\ldots,N-1;\; s=x,x+1,\ldots,N;\; y=1,\ldots,x-1\,;\; z=1,\ldots,N) \end{array}\right\} \quad (2,7)$$

Then the equations (2, 3) may be put in the following more condensed form

$$D_b H_{A_q}^{\nu}=\sum_{r=1}^{q+1} W_{b\,A_q}^{\;\;B_r} H_{B_r}^{\nu} \quad \left(q=1,\ldots,N,\; H_{B_{N+1}}^{\nu}=0,\; H_{B_1}^{\nu}=T_{B_1}^{\nu}\right) \quad (2,3e)$$

Theorem (2, 2). *The integrability conditions of* (2, 3) (*for* $m>1$) *are*

$$\left.\begin{array}{l} -\frac{1}{2}\,T_c^{\gamma}\,T_b^{\beta}\,R_{\gamma\beta\delta}^{\;\;\;\;\;\nu}\,H_{B_q}^{\delta}+\frac{1}{2}\,G_{cbB_q}^{\;\;\;\;A_q}\,H_{A_q}^{\nu}=H_{A_{q+2}}^{\nu}\,W_{[b\,B_q}^{\;\;\;C_{q+1}}\,W_{c]\,C_{q+1}}^{\;\;\;A_{q+2}}\;{}^{4)} \\ +\sum_{r=2}^{q+1} H_{A_r}^{\nu}\left(D_{[c}\,W_{b]\,B_q}^{\;\;\;A_r}+\sum_{s=r-1}^{q+1} W_{[b\,B_q}^{\;\;\;C_s}\,W_{c]\,C_s}^{\;\;\;A_r}\right) \\ \qquad\qquad +\,T_a^{\nu}\left(D_{[c}\,W_{b]\,B_q}^{\;\;\;a}+\sum_{s=1}^{q+1} W_{[b\,B_q}^{\;\;\;C_s}\,W_{c]\,C_s}^{\;\;\;a}\right) \\ \left(H_{B_1}^{\nu}=T_{B_1}^{\nu}\,,\;\; H_{B_{N+1}}^{\nu}=H_{B_{N+2}}^{\nu}=0\,,\;\; W_{b\,B_q}^{\;\;C_{N+1}}=0\,,\;\; q=1,\ldots,N\right) \end{array}\right\} \quad (2,8)$$

where

$$R_{\omega\mu\lambda}^{\;\;\;\;\;\;\nu}=\partial_{\mu}\,\Gamma_{\lambda\omega}^{\nu}-\partial_{\omega}\,\Gamma_{\lambda\mu}^{\nu}-\Gamma_{\alpha\omega}^{\nu}\,\Gamma_{\lambda\mu}^{\alpha}+\Gamma_{\alpha\mu}^{\nu}\,\Gamma_{\lambda\omega}^{\alpha}$$

is the curvature tensor of $\Gamma_{\alpha\beta}^{\gamma}$ *and*

$$G_{cbB_q}^{\;\;\;\;A_q}=\partial_b\,\Gamma_{B_q\,c}^{\;\;A_q}-\partial_c\,\Gamma_{B_q\,b}^{\;\;A_q}-\Gamma_{C_q\,c}^{\;\;A_q}\,\Gamma_{B_q\,b}^{\;\;C_q}+\Gamma_{C_q\,b}^{\;\;A_q}\,\Gamma_{B_q\,c}^{\;\;C_q}\,.$$

is the curvature tensor of $\Gamma_{B_q\,b}^{\;\;A_q}$.

Proof. From (2, 3e) we obtain

$$D_{[c}\,D_{b]}\,H_{A_q}^{\nu}=\sum_{r=1}^{q+1}\left[\left(D_{[c}\,W_{b]\,A_q}^{\;\;\;B_r}\right)H_{B_r}^{\nu}+W_{[b\,A_q}^{\;\;\;B_r}\,D_{c]}\,H_{B_r}^{\nu}\right]. \quad (2,9)$$

On the other hand we have obviously

$$2\,D_{[c}\,D_{b]}\,H_{A_q}^{\nu}=G_{cb\,A_q}^{\;\;\;\;B_q}\,H_{B_q}^{\nu}-R_{\gamma\beta\delta}^{\;\;\;\;\;\nu}\,H_{A_q}^{\delta}\,T_c^{\gamma}\,T_b^{\beta}\,.\;.\;. \quad (2.\,10)$$

The (necessary but not in general sufficient) conditions of integrability of (2, 3) (for $m>1$) are obtained by eliminating $D_b\,H_{B_r}^{\nu}$ and $D_{[c}\,D_{b]}\,H_{A_q}^{\nu}$ from (2, 3e), (2, 9) and (2, 10). In doing so, we obtain (2, 8).

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[4] We assume here and later on that the brackets *[ ] do not affect* capital indices. Thus for instance

$$W_{[b\,B_q}^{\;\;\;C_s}\,W_{c]\,C_s}^{\;\;\;A_r}=\frac{1}{2}\left(W_{b\,B_q}^{\;\;C_s}\,W_{c\,C_s}^{\;\;A_r}-W_{c\,B_q}^{\;\;C_s}\,W_{b\,C_s}^{\;\;A_r}\right).$$

3. *Consequences of integrability conditions.*

Let X_a^{ν} be a set of m linearly independent ξ-vectors, X_w^{ν} $(w = m^* + 1, \ldots, m^* + p \leqq n, m^* \geqq m)$ another set of p linearly independent ξ-vectors and finally Y_x^{ν} $(x = 1, \ldots, s,\ m + p + s < n)$ a set of s linearly independent ξ-vectors, the ξ-vectors X_a^{ν}, X_w^{ν}, Y_x^{ν} being linearly independent.

Definition (3, 1). *A domain D of A_n will be referred to as (m, m^*, p) general [as (m, m^*, p, s) general] if for any choice of the vectorfields X_a, X_w $[X_a, X_w, Y_x]$ there is at least one ξ-vectorfield Z_λ $[U_\lambda]$ incident with X_a, X_w $[X_a, X_w, Y_x]$ which satisfies the condition*

$$R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} X_a^{\omega} X_b^{\mu} X_c^{\lambda} Z_\nu \neq 0 \text{ in } D \quad . \quad . \quad . \quad . \quad . \quad (3, 1a)$$

$$[R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} X_a^{\omega} X_b^{\mu} Y_x^{\lambda} U_\nu \neq 0 \text{ in } D] \quad . \quad . \quad . \quad . \quad . \quad (3, 1b)$$

We use this definition in order to investigate the possibilities of imbedding a maximal A_m in an A_n for $m > 1$, $N > 1$. If m_s (n_s) is the dimension of the s-th osculating (normal) space $\overset{s}{E}_{m_s}$ $(\overset{s}{N}_{n_s})$ [4a] of an A_m, then the following theorem can be easily proved:

Theorem (3, 1). *A maximal A_m with $m_N < n$ $(m > 1)$ can not be imbedded in a $(m, m, m_N - m)$ general domain D of A_n.*

Proof. Let $\overset{1}{E}_m$ be spanned by the ξ-vectors X_a and let $\overset{N}{E}_{m_N} - \overset{1}{E}_m$ be spanned by the ξ-vectors X_w $\left(\text{hence } m^* = m,\ p = m_N - m = \sum_{1}^{N-1} {}_s\, n_s\right)$.

Because we are dealing with the case $m_N < n$, there is at least one ξ-vector Z_ν incident with X_a and X_w and any such vector is incident with T_a^{ν}, $H_{B_q}^{\nu}$ $(x = 2, \ldots, N)$. Hence we have by virtue of (2, 8) (for $q = 1$)

$$R_{\gamma\beta\delta}^{\cdot\cdot\cdot\nu} T_c^{\gamma} T_b^{\beta} T_d^{\delta} Z_\nu = 0$$

and this equation proves our theorem.

Theorem (3, 2). *A maximal A_m $(m > 1)$ with $N > 2$ can not be imbedded in a (m, m_2, n_2) general domain D of A_n.*

Proof. Let E_m $(\overset{2}{N}_{n_2})$ be spanned by the ξ-vectors X_a (X_w) [hence $m^* = m_2$, $p = n_2$]. Any arbitrary vector Z_ν incident with X_a, X_w is incident with T^{ν}_{a} and $H^{\nu}_{B_3}$. Hence we have by virtue of (2, 8) for $q=1$

[4a] In the maximal case the ξ-vectors H's do not overlap. If these ξ-vectors do not overlap, the dimensions of the osculating spaces and normal spaces are related by the equation (5, 8) of Theorem (5, 2) in *AI*, where the restriction about not overlapping H's was ommited. [There is an obvious misprint in this theorem, affecting the ξ-order of the H's and N's which is $(N + 1)$ and not (0) (through this order is (0) as far as only the Γ's are concerned).]

$$-R_{\gamma\beta\delta}{}^{\nu}\, T^{\gamma}_{c}\, T^{\beta}_{b}\, T^{\delta}_{B_1}\, Z_{\nu} = 2 H^{\nu}_{A_2}\, Z_{\nu} \left(D_{[c}\, W_{b]}{}^{A_2}_{B_1} + \sum_{1}^{2}{}_{r}\, W_{[b}{}^{C_r}_{B_1}\, W_{c]}{}^{A_2}_{C_r}\right). \quad (3, 2)$$

The coefficient in parenthesis ( ) vanishes for $N > 2$ because of (2, 7) and consequently (3, 2) reduces to

$$R_{\gamma\beta\delta}{}^{\nu}\, T^{\gamma}_{c}\, T^{\beta}_{b}\, T^{\delta}_{d}\, Z_{\nu} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (3, 3)$$

for any vector Z_{ν} incident with X_a ($= T_a$) and X_w. Hence D is not and can not be a (m, m_2, n_2) general domain.

This theorem can be easily generalized to

Theorem (3, 3). *A maximal A_m $(m > 1)$ with $N \geqq 4$ can not be imbedded in a $(m, m, m_N - m - n_q - n_{q-1}, n_{q-1})$ general domain D of A_n, $1 < q < N-1$.*

Proof. Let $\overset{1}{E}_m$, resp. $\overset{N}{E}_{m_N} - \overset{1}{E}_m - \overset{q}{N}_{n_q} - \overset{q-1}{N}_{n_{q-1}}$, resp. $\overset{q-1}{N}_{n_{q-1}}$ be spanned by X_a, resp. X_w, resp. Y_x (hence $m^* = m$, $p = m_N - m - n_q - n_{q-1}$, $s = n_{q-1}$, $m + p + s < n$). Any arbitrary ξ-vector U_{λ} incident with X_a, X_w, Y_x is incident with T^{ν}_{a}, $H^{\nu}_{B_u}$ $(u = 2, 3, \ldots, q-1, q, q+2, \ldots, N)$. Consequently we have from (2, 8)

$$\left.\begin{aligned} &-R_{\gamma\beta\alpha}{}^{\nu}\, T^{\gamma}_{c}\, T^{\beta}_{b}\, H^{\delta}_{B_q}\, U_{\nu} = \\ &= 2 H^{\nu}_{A_{q+1}}\, U_{\nu} \left(D_{[c}\, W_{b]}{}^{A_{q+1}}_{B_q} + \sum_{q}^{q+1}{}_{s}\, W_{[b}{}^{C_s}_{B_q}\, W_{c]}{}^{A_{q+1}}_{C_s}\right) \end{aligned}\right\} \quad . \; . \quad (3, 4)$$

The coefficient in parenthesis ( ) vanishes for $q < N - 1$ because of (2, 7) and consequently (3, 4) reduces to

$$R_{\gamma\beta\delta}{}^{\nu}\, T^{\gamma}_{c}\, T^{\beta}_{b}\, H^{\delta}_{B_q}\, U_{\nu} = 0$$

and this equation is equivalent to

$$R_{\gamma\beta\delta}{}^{\nu}\, X^{\gamma}_{c}\, X^{\beta}_{b}\, Y^{\delta}_{x}\, U_{\nu} = 0$$

which proves our theorem.

Theorem (3, 4). *A maximal A_m $(m > 1)$ with $N > 2$ can not be imbedded in a flat A_n.*

Proof. The equation (2, 8) reduces for $q = 1$ and $N > 2$ in a flat A_n to

$$\tfrac{1}{2} G_{cbd}{}^{e}\, T^{\nu}_{e} = \delta_{[b}{}^{C_2}_{|d|}\, \delta_{c]}{}^{A_3}_{C_2}\, H^{\nu}_{A_3} + \delta_{[b|d|}{}^{C_2}\, K_{c]}{}^{e}_{C_2}\, T^{\nu}_{e}.$$

Because $H^{\nu}_{A_3}$ is in $\overset{2}{N}_{n_2}$ and T^{ν}_{a} in $\overset{1}{E}_m$, this equation splits in two,

$$\tfrac{1}{2} G_{cbd}{}^{e} = \delta_{[b}{}^{C_2}_{|d|}\, K_{c]}{}^{e}_{C_2} \equiv K_{[c\, b]}{}^{e}_{d}$$

$$\delta_{[b}{}^{C_2}_{|d|}\, \delta_{c]}{}^{A_3}_{C_2}\, H^{\nu}_{A_3} \equiv H^{\nu}_{[c\, b]\, d} = 0.$$

The last equation shows that the dimension of the space N_{n_3} spanned by $\overset{\nu}{H}_{bcd}$ never can reach $\frac{m^2(m+1)}{2}$. Hence the dimension of the space $\overset{3}{E}_{m_3}$ never can reach $m + \frac{m(m+1)}{2} + \frac{m^2(m+1)}{2}$ and consequently, our A_m can not be a maximal one (cf. the introduction in *AI*).

4. *Auxiliary results.*

In this section some auxiliary results will be established which we shall need later on. Let us introduce a set of functions

$$\Phi_{B_p a}^{\;\;\;\;\;C_p} \equiv \Gamma_{B_p a}^{\;\;\;\;\;C_p} + W_{a B_p}^{\;\;\;\;\;C_p}\;{}^{5)} \quad \text{and} \quad \Phi_{B_p a}^{\;\;\;\;\;C_q} \equiv W_{a B_p}^{\;\;\;\;\;C_q}\;{}^{5)}$$

$$p, q = 1, \ldots, N;\; p \neq q.$$

These functions may be thought of as coefficients of a connection which will be referred to as a *Frenet connection*. If $A, B, C, \ldots$ run from 1 to m^N, then a set of condensed ξ-vectors $\overset{\nu}{V}_{A_p}$ may be thought of as a set of components of $\overset{\nu}{V}_A$ which we call again a condensed ξ-vector. If F_e is the covariant derivate operator of the FRENET connection which operates also on ξ-tensors (by means of ∇_e) then the equation

$$F_e \overset{\nu}{V}_A \equiv \nabla_e \overset{\nu}{V}_A - \Phi_{Ae}^{\;\;\;B} \overset{\nu}{V}_B$$

is equivalent to

$$\left.\begin{aligned} F_e \overset{\nu}{V}_{A_p} &= \nabla_e \overset{\nu}{V}_{A_p} - \sum_{1}^{N}{}_r\, \Phi_{A_p e}^{\;\;\;\;\;B_r} \overset{\nu}{V}_{B_r} = \\ &= D_e \overset{\nu}{V}_{A_p} - \sum_{1}^{N}{}_q\, W_{e A_p}^{\;\;\;\;\;B_q} \overset{\nu}{V}_{B_q}\;{}^{6)} \quad , \quad p = 1, \ldots, N. \end{aligned}\right\} \quad (4,1)$$

Lemma (4, 1). *The Frenet formulae* (2, 3) *are equivalent to*

$$F_e \overset{\nu}{H}_{A_p} = 0 \quad , \quad p = 1, \ldots, N. \quad . \quad . \quad . \quad . \quad . \qquad (4,2)$$

The proof follows at once (4, 1) and (2, 3*e*).

Note. If w_μ is an arbitrary vector-field A_m and $w_A = \overset{\nu}{H}_A w_\nu$, then

$$F_e w_{A_p} = F_e \overset{\nu}{H}_{A_p} w_\nu = \overset{\nu}{H}_{A_p} F_e w_\nu = \overset{\nu}{H}_{A_p} \nabla_e w_\nu .$$

Hence in studying $\nabla_e w_\nu$, we could confine ourselves to its „canonical form",

$$F_e w_{A_p} = D_e w_{A_p} - \sum_{1}^{N}{}_q\, W_{e A_p}^{\;\;\;\;\;B_q} w_{B_q} .$$

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5) Cf. (1,6) and (2,7).

6) According to (2,7) we have $W_{a B_p}^{\;\;\;\;\;C_r} = 0$ for $p < N$ and $r = p, p+2, \ldots, N$.

Let us introduce the mixed $(\xi\eta)$-tensor

$$\left.\begin{aligned}\text{a)}\quad \overset{0}{I}{}_{eb}{}^{\nu}{}_{B_q} = {} & H^{\nu}_{A_{q+2}} W_{[b}{}^{C_{q+1}}_{B_q} W_{e]}{}^{A_{q+2}}_{C_{q+1}} + \\ & + \sum_{r=2}^{q+1} H^{\nu}_{A_r}\Big(D_{[e} W_{b]}{}^{A_1}_{B_q} + \sum_{s=r-1}^{q+1} W_{[b}{}^{C_s}_{B_q} W_{e]}{}^{A_r}_{C_s} - \frac{\delta_{rq}}{2} G_{eb}{}^{A_r}_{B_q}\Big) + \\ & + T^{\nu}_{A_1}\Big(D_{[e} W_{b]}{}^{A_1}_{B_q} + \sum_{s=1}^{q+1} W_{[b}{}^{C_s}_{B_q} W_{e]}{}^{A_1}_{C_s} - \frac{\delta_{1q}}{2} G_{eb}{}^{A_1}_{B_q}\Big), \; (q=1,\ldots,N)\end{aligned}\right\} \tag{4, 3}$$

and

$$\overset{0}{Q}{}_{CBA}{}^{\nu} \equiv -\tfrac{1}{2} H^{\omega\mu\lambda}_{CBA} R_{\omega\mu\lambda}{}^{\nu} \quad \ldots\ldots \tag{4, 3b}$$

$$\left.\begin{aligned}\overset{s}{Q}{}_{F\ldots DCBA}{}^{\nu} \equiv -\tfrac{1}{2} H^{\alpha_s\ldots\alpha_1\omega\mu\lambda}_{F\ldots DCBA} \nabla_{\alpha_s}\ldots\nabla_{\alpha_1} R_{\omega\mu\lambda}{}^{\nu}, \\ \left(H^{\alpha_p\ldots\alpha_1}_{F\ldots D} = H^{\alpha_p}_{F}\ldots H^{\alpha_1}_{D}\right).\end{aligned}\right\} \quad \ldots \tag{4, 3c}$$

Then the following lemma may be easily proved:

Lemma (4, 2). *The following equations hold*

$$\left.\begin{aligned}\text{a)}\quad & \overset{0}{Q}{}_{ebB_q}{}^{\nu} = \overset{0}{I}{}_{ebB_q}{}^{\nu} \qquad (q=1,\ldots,N) \\ \text{b)}\quad & \overset{s}{Q}{}_{e_s\ldots e_1 cbA}{}^{\nu} = F_{e_s} \overset{s-1}{Q}{}_{e_{s-1}\ldots e_1 cbA}{}^{\nu}, \quad s=1,2,\ldots\end{aligned}\right\} \quad \ldots \tag{4, 4}$$

Proof. The equation (4, 4a) is obviously equivalent to (2, 8). From (4, 3bc) we have, on account of (4, 2)

$$F_e \overset{0}{Q}{}_{CBA}{}^{\nu} = -\tfrac{1}{2} H^{\omega\mu\lambda}_{CBA} F_e R_{\omega\mu\lambda}{}^{\nu} = -\tfrac{1}{2} H^{\omega\mu\lambda\alpha}_{CBAe} \nabla_{\alpha} R_{\omega\mu\lambda}{}^{\nu} = \overset{1}{Q}{}_{eCBA}{}^{\nu}.$$

The remaining equations of (4, 4*b*) can be proved in a similar manner.

Let us denote by $\overset{s}{I}$ the condensed $(\xi\eta)$-tensor $\overset{s}{Q}$, $s=1,2,\ldots$, where $\overset{0}{Q}$ is replaced by $\overset{0}{I}$ (according to (4, 4a). $\overset{s}{Q}$ depends on $R_{\omega\mu\lambda}{}^{\nu}$ and its first s derivatives. We use $\overset{s}{I}$ for a construction of a mixed $(\xi\eta)$-tensor which depends on R and its s—1 first derivatives only.

Lemma (4, 3). *Put*

$$\overset{s}{J}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} \equiv D_{e_{s+2}} \overset{s-1}{I}{}_{e_{s+1}\ldots e_1}{}^{\nu}{}_{A_p} - \sum_{q=1}^{N} W_{e_{s+2}}{}^{B_q}_{A_p} \overset{s-1}{I}{}_{e_{s+1}\ldots e_1}{}^{\nu}{}_{B_q}. \tag{4, 5a}$$

Then

$$\overset{s}{J}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} = \overset{s}{I}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} + \sum_{k=1}^{s+1} \delta_{e_{s+2}}{}^{E_2}_{e_k} \overset{s-1}{Q}{}_{e_{s+1}\ldots e_{k+1} E_2 e_{k-1}\ldots e_1}{}^{\nu}{}_{A_p}. \tag{4, 5b}$$

Moreover $\overset{1}{J}$ *does not contain* $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ *at all and* $\overset{s}{J}$ *for* $s=2,3,\ldots$ *depends on* $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ *and its first* $s-1$ *covariant derivatives only,*

Proof. We have from (4, 4*b*) for $s=1,2,\ldots$

$$\left.\begin{aligned}\overset{s}{Q}{}_{e_{s+2}\ldots e_1 A_p}^{\cdot\cdot\cdot\cdot\nu} = D_{e_{s+2}}\overset{s-1}{Q}{}_{e_{s+1}\ldots e_1 A_p}^{\cdot\cdot\cdot\cdot\nu} - \sum_{1}^{N}{}_{q}\, W_{e_{s+2}}{}^{B_q}_{A_p}\overset{s-1}{Q}{}_{e_{s+1}\ldots e_1 B_q}^{\cdot\cdot\cdot\cdot\nu} \\ - \sum_{1}^{s+1}{}_{k}\, \delta_{e_{s+2}\,e_k}^{E_2}\overset{s-1}{Q}{}_{e_{s+1}\ldots e_{k+1}E_2 e_{k-1}\ldots e_1 A_p}^{\cdot\cdot\cdot\cdot\nu}\end{aligned}\right\} \quad (4,6)$$

and this equation proves (4, 5b). Moreover the right hand member of (4, 5a) for $s=1$ does not contain $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ at all. According to (4, 6) for $s=1$, $\overset{1}{I}{}_{e_3 e_2 e_1 A_p}^{\cdot\cdot\cdot\cdot\nu}$ contains $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ and does not contain its derivatives. Hence we see from (4, 5a) for $s=2$ that $\overset{2}{J}{}_{e_4\ldots e_1 A_p}^{\cdot\cdot\cdot\cdot\nu}$ depends on $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ and its first derivative only. In a similar manner all the remaining statements of our lemma may be proved.

5. *External contact invariants.*

Let A_m and $'A_m$ $(m>1)$ be two subspaces referred to the same parameter system and having a point P at common. All objects belonging to $'A_m$ will be designated by the same letters but for the primes as the corresponding objects of A_m. We shall assume $N='N$.

Definition (5, 1). A_m *and* $'A_m$ *are said to have an external contact of order* $r=1,2,\ldots,N$ *if*

$$H_{A_q}^{\cdot\nu}={}'H_{A_q}^{\cdot\nu} \text{ at } P, \qquad q=1,2,\ldots,r.\,^{7)} \quad \ldots \quad (5,1)$$

Theorem (5, 1). *If* A_m *and* $'A_m$ *have external contact of order* $r=1$ *then we have*

$$\overset{0}{I}{}_{cba}^{\cdot\cdot\cdot\nu}={}'\overset{0}{I}{}_{cba}^{\cdot\cdot\cdot\nu} \text{ at } P. \quad \ldots\ldots\ldots \quad (5,2)$$

Proof. Because of (5, 1) for $r=1$ we have from (4, 3*b*) and from a similar equation for $'A_m$

$$\overset{0}{Q}{}_{cba}^{\cdot\cdot\cdot\nu}={}'\overset{0}{Q}{}_{cba}^{\cdot\cdot\cdot\nu} \text{ at } P. \quad \ldots\ldots\ldots \quad (5,3)$$

The equation (5, 2) follows at once from (5, 3) and (4, 4*a*).

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[7] We put here and later on

$$H_{A_1}^{\cdot\nu}=T_{A_1}^{\cdot\nu} \quad , \quad 'H_{A_1}^{\cdot\nu}={}'T_{A_1}^{\cdot\nu}$$

(cf. the footnote [3])).

Theorem (5, 2). *If A_m and $'A_m$ have external contact of order $r\,(=1, 2, 3, \ldots, N)$ at P, then we have*

$$\left.\begin{array}{ll}\text{a)} & \overset{0}{I}{}_{cb}{}^{\nu}{}_{A_p} = {}'\overset{0}{I}{}_{cb}{}^{\nu}{}_{A_p} \text{ at } P, \qquad p=1, 2, \ldots, r \\ \text{b)} & \overset{1}{J}{}_{dcb}{}^{\nu}{}_{A_p} = {}'\overset{1}{J}{}_{dcb}{}^{\nu}{}_{A_p} \text{ at } P, \qquad p=1, 2, \ldots r \\ \text{c)} & \overset{s}{J}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} = {}'\overset{s}{J}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} \text{ at } P, \quad p=1,2,\ldots,r;\ s=2,3,\ldots \end{array}\right\} \quad (5,4)$$

The first two equations do not involve $R_{\omega\mu\lambda}{}^{\nu}$. The last one contains $R_{\omega\mu\lambda}{}^{\nu}$ and its first $s-1$ covariant derivatives.

Proof. If A_m and $'A_m$ have external contact of order r at P then we have from (5, 1), from (4, 3*b*, *c*) and from similar equations for $'A_m$

$$\overset{s}{Q}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} = {}'\overset{s}{Q}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} \text{ at } P, \quad s=0, 1, \ldots;\ p=1, \ldots, r. \qquad (5,5)$$

From (5, 5) for $s=0$, from (4, 4a) and from a similar equation for $'A_m$ we get (5, 4a) and this equation does not contain $R_{\omega\mu\lambda}{}^{\nu}$ at all. On the other hand (5, 5) for $s=1, 2, \ldots$ is equivalent to

$$\overset{s}{I}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} = {}'\overset{s}{I}{}_{e_{s+2}\ldots e_1}{}^{\nu}{}_{A_p} \text{ at } P, \quad s=1, 2, \ldots;\ p=1, 2, \ldots, r. \qquad (5,6)$$

The equations (5, 4*b*, *c*) follow at once from (5, 6), from (4, 3c), (4, 5*b*) and similar equations for $'A_m$. The remaining statements follow from the properties of the J's stated in the lemma (4, 3).

Note. The theorems (5, 1) and (5, 2) hold also for a flat A_n but are obviously trivial in this case. On the other hand,

$$\overset{0}{I}{}_{cb}{}^{\nu}{}_{A_p} = 0 \quad \text{along } A_m, \quad p=1, \ldots, N$$

represents the *fundamental Gauss-Coddazi-Kühne equations* for an A_m in a flat A_n.

Indiana University
Department of Mathematics

Bloomington (Ind.) USA

Mathematics. — *On the evaluation of certain integrals occurring in the theory of the freely vibrating circular disk and related problems.* By C. J. BOUWKAMP. (Philips Research Laboratories, Eindhoven, Netherlands.) (Communicated by Prof. H. B. G. CASIMIR.)

(Communicated at the meeting of September 24, 1949.)

In studying the radiation of sound by a freely vibrating, rigid, circular disk for low frequencies, I recently came across the repeated integral

$$I_{n,m}(\varrho) = \int_0^1 P_{2n+1}(\sqrt{1-\varrho'^2})\,\varrho'\,d\varrho'\,\frac{1}{2\pi}\int_0^{2\pi}[\varrho^2-2\varrho\varrho'\cos\vartheta'+\varrho'^2]^{\frac{1}{2}(m-1)}\,d\vartheta', \quad (1)$$

in which n and m are non-negative integers, P is the Legendre polynomial, and $0 \leq \varrho \leq 1$. The same integral is encountered in the theory of diffraction of plane waves by circular disks and apertures. The purpose of this paper is to evaluate the integral (1) in terms of known functions and to derive a number of relations that are useful for the physical applications referred to, details of which will be published in Physica.

First of all it will be shown that $I_{n,m}(\varrho)$ can be obtained from $I_{n,m+2}(\varrho)$ by means of differentiation, namely,

$$\mathrm{D}\, I_{n,m+2}(\varrho) = (m+1)^2\, I_{n,m}(\varrho), \quad \dots\dots\dots \quad (2)$$

in which D is the differential operator

$$\mathrm{D} = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial}{\partial\varrho}\right). \quad \dots\dots\dots \quad (3)$$

Equation (2) even holds if the function $P_{2n+1}(\sqrt{1-\varrho'^2})$ occurring in (1) is replaced by any function $\varphi_n(\varrho')$ for which the corresponding integrals exist. To prove this, let us consider the *wave function* $\psi(z, \varrho)$ defined by

$$\psi(z,\varrho) = \iint_S \varphi(\varrho')\,\frac{e^{ikr}}{r}\,dS, \quad \dots\dots\dots \quad (4)$$

in which z, ϱ, ϑ are cylindrical coordinates around the axis of the disk S of unit radius. The domain of integration is given by $z' = 0$, $0 \leq \varrho' \leq 1$, $0 \leq \vartheta' \leq 2\pi$, while r denotes the distance between the point of integration $(0, \varrho', \vartheta')$ and the field point $(z, \varrho, 0)$; k is the wave number. This function (4) is annihilated by the operator $\mathrm{D} + k^2 + \partial^2/\partial z^2$, because it is an axially symmetric solution of the wave equation, $\Delta\psi + k^2\psi = 0$. Let us apply this operator to the power-series expansion of ψ with respect to ik, viz.,

$$(\mathrm{D} + k^2 + \partial^2/\partial z^2)\sum_{\nu=0}^{\infty}\{(ik)^\nu/\nu!\}\iint_S \varphi(\varrho')\,r^{\nu-1}\,dS = 0. \quad \dots \quad (5)$$

Arranging the left-hand member according to powers of ik, we are led to a power series that has to vanish identically. Therefore all coefficients have to vanish; this gives

$$(\mathrm{D} + \partial^2/\partial z^2) \iint_S \varphi(\varrho')\, r^{-1}\, \mathrm{d}S = 0, \quad \ldots \ldots \quad (6)$$

$$\mathrm{D} \iint_S \varphi(\varrho')\, \mathrm{d}S = 0, \quad \ldots \ldots \quad (7)$$

$$\frac{1}{(\nu-2)!} \iint_S \varphi(\varrho')\, r^{\nu-3}\, \mathrm{d}S = \frac{1}{\nu!} (\mathrm{D} + \partial^2/\partial z^2) \iint_S \varphi(\varrho')\, r^{\nu-1}\, \mathrm{d}S. \quad . \quad (8)$$
$$(\nu = 2, 3, \ldots)$$

We now perform the differentiation with respect to z in (8) under the signs of integration, using the identity

$$(\partial^2/\partial z^2)\, r^{\nu-1} = (\nu-1)\, r^{\nu-3} + (\nu-1)(\nu-3)\, z^2 r^{\nu-5}, \quad \ldots \quad (9)$$

and we consider the limiting case $z \to 0$ $(0 \leq \varrho \leq 1)$. As is easily seen, the term with z^2 in (9) does not contribute to the limit because the integral $\iint \varphi(\varrho') r^{\nu-5} \mathrm{d}S$, in the most unfavourable case $(\nu = 2)$, is only of the order $1/z$. In the limit $z \to 0$ we can replace r by

$$s = \{\varrho^2 - 2\varrho\varrho' \cos\vartheta' + \varrho'^2\}^{\frac{1}{2}}, \quad \ldots \ldots \quad (10)$$

so that (8) becomes, for $z \to 0$,

$$\nu(\nu-1) \iint_S \varphi(\varrho')\, s^{\nu-3}\, \mathrm{d}S = \mathrm{D} \iint_S \varphi(\varrho')\, s^{\nu-1}\, \mathrm{d}S + (\nu-1) \iint_S \varphi(\varrho')\, s^{\nu-3}\, \mathrm{d}S,$$

or

$$\mathrm{D} \iint_S \varphi(\varrho')\, s^{\nu-1}\, \mathrm{d}S = (\nu-1)^2 \iint_S \varphi(\varrho')\, s^{\nu-3}\, \mathrm{d}S. \quad \ldots \quad (11)$$

Equation (2) is included in (11) for $2\pi\varphi(\varrho') = P_{2n+1}(\sqrt{1-\varrho'^2})$ and $\nu = m + 2$. In passing it may be noticed that eq. (7) is trivial, the integral being independent of ϱ. Furthermore, it follows from (6) that

$$\lim_{z \to +0} -(\partial^2/\partial z^2) \iint_S \varphi(\varrho')\, r^{-1}\, \mathrm{d}S = \mathrm{D} \iint_S \varphi(\varrho')\, s^{-1}\, \mathrm{d}S. \quad \ldots \quad (12)$$

In particular we thus get

$$\lim_{z \to +0} -(1/2\pi)(\partial^2/\partial z^2) \iint_S P_{2n+1}(\sqrt{1-\varrho'^2})\, r^{-1}\, \mathrm{d}S = \mathrm{D}\, I_{n,0}(\varrho). \quad . \quad (13)$$

Hence, if we *define* the function $I_{n,-2}(\varrho)$ by the relation

$$I_{n,-2}(\varrho) = \lim_{z \to +0} -(1/2\pi)(\partial^2/\partial z^2) \iint_S P_{2n+1}(\sqrt{1-\varrho'^2})\, r^{-1}\, \mathrm{d}S, \quad . \quad (14)$$

then eq. (2) holds even for $m = -2$. In what follows, however, m will be a non-negative integer unless otherwise stated.

As a first step towards the evaluation of $I_{n,m}(\varrho)$ let us consider this function at $\varrho = 0$. We have

$$I_{n,m}(0) = \int_0^1 P_{2n+1}(\sqrt{1-\varrho'^2})\,\varrho'^m\,d\varrho' = \int_0^{\pi/2} P_{2n+1}(\cos\vartheta)\sin^m\vartheta\cos\vartheta\,d\vartheta. \qquad (15)$$

Now the Legendre polynomials can be expressed in terms of hypergeometric functions:

$$P_{2n+1}(\cos\vartheta) = (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2}\cos\vartheta\, F(-n, n+\tfrac{3}{2}; \tfrac{3}{2}; \cos^2\vartheta).$$

If this finite series is substituted in (15) we can integrate term by term by employing well-known expressions for the trigonometric integrals that occur. The result becomes

$$I_{n,m}(0) = (-1)^n \frac{(2n+1)!}{2^{2n+1}(n!)^2}\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}m+2)} F(-n, n+\tfrac{3}{2}; \tfrac{1}{2}m+2; 1).$$

The remaining hypergeometric series can be summed by Gauss's formula, and after some transformation the final result is found to be

$$I_{n,m}(0 = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2}m+n+2)\,\Gamma(\frac{1}{2}m-n+\frac{1}{2})}. \quad . \quad (16)$$

In the second place we shall determine the value of $I_{n,0}(\varrho)$. For that purpose let us consider the *potential function*

$$V = (1/2\pi)\iint_S \varphi\, r^{-1}\,dS, \qquad \varphi = P_{2n+1}(\sqrt{1-\varrho'^2}), \quad . \quad . \quad . \quad (17)$$

V being a solution of $\triangle V = 0$. This potential function is most simply expressed in terms of oblate-spheroidal potential functions. The corresponding oblate-spheroidal coordinates ξ, η, ϑ adapted to the circular disk (on which $\eta = 0$) can be defined in terms of the cylindrical coordinates by $z = \xi\eta$, $\varrho^2 = (1-\xi^2)(1+\eta^2)$, $(-1 \leq \xi \leq 1)$, $(\eta \geq 0)$, while the angular variable is the same in the two systems. Now the characteristic solutions of the potential equation, for axially symmetric exterior problems, are provided by $P_n(\xi)Q_n(i\eta)$, where Q denotes the Legendre function of the second kind. Moreover, since V is an even function of z, only terms of even degree occur in V. Hence we can write, with as yet unknown constants λ_ν,

$$V = \sum_{\nu=0}^{\infty} i\lambda_\nu P_{2\nu}(\xi)\,Q_{2\nu}(i\eta).$$

The coefficients in this expansion are easy to calculate for any given function φ; recalling the connection between the electric field strength and the charge density in the theory of electricity, we have

$$\varphi(\varrho') = \lim_{z\to+0} -\frac{\partial V}{\partial z} = \lim_{\eta\to 0} -\frac{1}{\xi}\frac{\partial V}{\partial \eta}. \qquad (0 \leq \varrho' \leq 1)$$

On the positive side ($z = +0$) of the disk S one has $\xi = +(1 - \varrho^2)^{\frac{1}{2}}$, $\eta = +0$. Consequently, in the case of (17), λ_ν must be determined from

$$\xi\, P_{2n+1}(\xi) = \sum_{\nu=0}^{\infty} \lambda_\nu\, Q_{2\nu}'(+i\,0)\, P_{2\nu}(\xi) = \sum_{\nu=0}^{\infty} \lambda_\nu\, P_{2\nu}(\xi)/P_{2\nu}(0).$$

The left-hand member of this equation can be expressed as a linear combination of $P_{2n}(\xi)$ and $P_{2n+2}(\xi)$; thus all λ's are zero except two of them. These exceptions are

$$\lambda_n = \frac{2n+1}{4n+3}\, P_{2n}(0), \quad \lambda_{n+1} = \frac{2n+2}{4n+3}\, P_{2n+2}(0).$$

However, we only need the value of V for $z = +0$, $0 \leq \varrho \leq 1$. This value is given by

$$V_0 = \sum_{\nu=0}^{\infty} i\, \lambda_\nu\, P_{2\nu}(\xi)\, Q_{2\nu}(+i\,0) = \frac{\pi}{2} \sum_{\nu=0}^{\infty} \lambda_\nu\, P_{2\nu}(0)\, P_{2\nu}(\xi).$$

This in fact is an explicit expression for $I_{n,0}(\varrho)$; after some transformation we find that

$$\left.\begin{aligned} I_{n,0}(\varrho) &= \frac{\Gamma(n+\frac{1}{2})\, \Gamma(n+\frac{3}{2})}{2\, \Gamma(n+1)\, \Gamma(n+2)} \times \\ &\times \left[\frac{2n+2}{4n+3}\, P_{2n}(\sqrt{1-\varrho^2}) + \frac{2n+1}{4n+3}\, P_{2n+2}(\sqrt{1-\varrho^2})\right]. \end{aligned}\right\} \quad . \tag{18}$$

An alternative expression for $I_{n,0}(\varrho)$ is obtained when the Legendre polynomials in (18) are expressed as hypergeometric functions. In this connection we have

$$P_{2n+1}(\sqrt{1-\varrho^2}) = F(-n, n+\tfrac{1}{2}; 1; \varrho^2), \quad . \quad . \quad . \quad . \tag{19}$$

from which follows

$$\frac{2n+2}{4n+3}\, P_{2n}(\sqrt{1-\varrho^2}) + \frac{2n+1}{4n+3}\, P_{2n+2}(\sqrt{1-\varrho^2}) = F(-n-1, n+\tfrac{1}{2}; 1; \varrho^2). \tag{20}$$

If this is substituted in (18) we obtain after some transformation that

$$I_{n,0}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\, \Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})}{\Gamma(n+1)\, \Gamma(n+2)\, \Gamma(-n+\frac{1}{2})}\, F(-n-1, n+\tfrac{1}{2}; 1; \varrho^2). \tag{21}$$

It should be noted that eqs (16) and (21) are consistent with each other.

We are also able to evaluate the function $I_{n,-2}(\varrho)$ in virtue of

$$I_{n,-2}(\varrho) = \mathrm{D}\, I_{n,0}(\varrho) = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial}{\partial \varrho}\right) I_{n,0}(\varrho). \quad . \quad . \quad . \quad . \tag{22}$$

Two different expressions can be derived, according as (18) or (21) is employed. The first leads to

$$I_{n,-2}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\, \Gamma(-\frac{1}{2})\, \Gamma(-\frac{1}{2})}{\Gamma(n+1)\, \Gamma(n+1)\, \Gamma(-n-\frac{1}{2})}\, \frac{P_{2n+1}(\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}}, \tag{23}$$

as is readily verified by using well-known properties of Legendre polynomials. On the other hand eq. (21) yields

$$I_{n,-2}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(-\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(n+1)\,\Gamma(n+1)\,\Gamma(-n-\frac{1}{2})} F(-n, n+\tfrac{3}{2}; 1; \varrho^2), \quad (24)$$

which is from (21) most easily obtained with the aid of

$$\mathrm{D}\, F(a, b; 1; \varrho^2) = 4ab\, F(a+1, b+1; 1; \varrho^2). \quad . \quad . \quad . \quad (25)$$

The agreement between (23) and (24) follows from the known identity

$$\frac{P_{2n+1}(\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}} = F(-n, n+\tfrac{3}{2}; 1; \varrho^2). \quad . \quad . \quad . \quad . \quad (26)$$

The general expression for $I_{n,m}(\varrho)$ can now be written down almost immediately by simply inspecting (16), (21), and (24). First of all it will be observed that the numerical factors occurring in (21) and (24) are the respective values of $I_{n,m}(0)$, generally given by (16). We thus come to the conjecture that

$$I_{n,m}(\varrho) = I_{n,m}(0)\;\; F(A, B; 1; \varrho^2),$$

in which A and B depend on n and m. Further, $A = -n-1$ for $m = 0$, and $A = -n$ for $m = -2$. It is thus reasonable to expect that $A = -\frac{1}{2}(2n+m+2)$ for general values of n and m. By a similar argument we expect to have $B = \frac{1}{2}(2n-m+1)$. In other words, our final conjecture is that

$$\left.\begin{aligned} I_{n,m}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2}m+n+2)\,\Gamma(\frac{1}{2}m-n+\frac{1}{2})} \times \\ \times F(-n-\tfrac{1}{2}m-1, n-\tfrac{1}{2}m+\tfrac{1}{2}; 1; \varrho^2), \end{aligned}\right\} \quad (27)$$

for all non-negative integers n and m. The truth of this conjecture can be proved as follows. (i) Equation (27) is correct for $m = 0$ and $n = 0, 1, 2, \ldots$, as is obvious from (21). (ii) Equation (27) is correct for $m = 1$ and $n = 0, 1, 2, \ldots$: it follows from (1) and (16) that $I_{n,1}(\varrho) = I_{n,1}(0) = \frac{1}{3}\,(n = 0)$ or $0\,(n > 0)$, which values are also given by (27). (iii) The right-hand member of (27) satisfies (2); this is readily verified with the aid of (25). (iv) If the function $I_{n,m}(\varrho)$ is known for some value of m, then $I_{n,m+2}(\varrho)$ can be obtained by integration of the inhomogeneous differential equation (2) under suitable initial conditions. Now the difference of any two solutions of (2), regarded as an equation for $I_{n,m+2}(\varrho)$ while $I_{n,m}(\varrho)$ is given, is at the utmost equal to $y = C_1 \log \varrho + C_2$, this function being the general solution of the corresponding homogeneous differential equation, $\mathrm{D}y = 0$. The first constant of integration, C_1, may be taken equal to zero because all the integrals considered are finite at $\varrho = 0$. The constant C_2 is uniquely determined by the value of $I_{n,m+2}(\varrho)$ at $\varrho = 0$, and this initial value is correctly given by (27). This completes the proof of eq. (27). As a corollary, it follows from the reasoning under (iv) that

$$I_{n,m+2}(\varrho) - I_{n,m+2}(0) = (m+1)^2 \int_0^{\varrho} u^{-1}\,\mathrm{d}u \int_0^{u} I_{n,m}(v)\, v\,\mathrm{d}v,$$

or, after integration by parts,

$$I_{n,m+2}(\varrho) = I_{n,m+2}(0) - (m+1)^2 \varrho^2 \int_0^1 t\, I_{n,m}(\varrho t) \log t\,\mathrm{d}t, \quad . \quad (28)$$

which is the counterpart of eq. (2).

To simplify the discussion of the function $I_{n,m}(\varrho)$ we consider even and odd values of m separately. In the latter case we have

$$\left.\begin{aligned} I_{n,2m+1}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(m+1)\,\Gamma(m+1)}{\Gamma(n+1)\,\Gamma(m+n+\frac{5}{2})\,\Gamma(m-n+1)} \times \\ \times F(-n-m-\tfrac{3}{2}, n-m; 1; \varrho^2). \end{aligned}\right\} \quad (29)$$

This is identically zero if $0 \leq m < n$, because of the last gamma function in the denominator. On the other hand if $m \geq n \geq 0$ the right-hand member is a polynomial in ϱ^2 of degree $m-n$, since in this case the second parameter of the hypergeometric series is the non-positive integer $n-m$. Furthermore, in terms of Jacobi polynomials we have

$$F(-n-m-\tfrac{3}{2}, n-m; 1; \varrho^2) = G_{m-n}(-2m-\tfrac{3}{2}, 1, \varrho^2). \quad . \quad (30)$$

For even values of the second subscript we get

$$\left.\begin{aligned} I_{n,2m}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(m+n+2)\,\Gamma(m-n+\frac{1}{2})} \times \\ \times F(-n-m-1, n-m+\tfrac{1}{2}; 1; \varrho^2), \end{aligned}\right\} \quad (31)$$

which is always a polynomial in ϱ^2 of degree $n+m+1$. In terms of Jacobi polynomials one has

$$F(-n-m-1, n-m+\tfrac{1}{2}; 1; \varrho^2) = G_{n+m+1}(-2m-\tfrac{1}{2}, 1, \varrho^2). \quad (32)$$

The first few functions are listed below:

$$I_{0,1}(\varrho) = \frac{1}{3};\ I_{n,1}(\varrho) = 0 \ \ (n > 0);$$

$$I_{0,3}(\varrho) = \frac{2}{15}\left(1 + \frac{5}{2}\varrho^2\right);\ I_{1,3}(\varrho) = -\frac{2}{35};\ I_{n,3}(\varrho) = 0 \ (n > 1).$$

$$I_{0,0}(\varrho) = \frac{\pi}{4}\left(1 - \frac{1}{2}\varrho^2\right);\ I_{1,0}(\varrho) = \frac{3\pi}{32}\left(1 - 3\varrho^2 + \frac{15}{8}\varrho^4\right);$$

$$I_{2,0}(\varrho) = \frac{15\pi}{256}\left(1 - \frac{15}{2}\varrho^2 + \frac{105}{8}\varrho^4 - \frac{105}{16}\varrho^6\right);$$

$$I_{3,0}(\varrho) = \frac{175\pi}{4096}\left(1 - 14\varrho^2 + \frac{189}{4}\varrho^4 - \frac{231}{4}\varrho^6 + \frac{3003}{128}\varrho^8\right);$$

$$I_{0,2}(\varrho) = \frac{\pi}{16}\left(1 + \varrho^2 - \frac{1}{8}\varrho^4\right);$$

$$I_{1,2}(\varrho) = -\frac{\pi}{64}\left(1 - \frac{3}{2}\varrho^2 + \frac{9}{8}\varrho^4 - \frac{5}{16}\varrho^6\right);$$

$$I_{2,2}(\varrho) = -\frac{5\pi}{2048}\left(1 - 6\varrho^2 + \frac{45}{4}\varrho^4 - \frac{35}{4}\varrho^6 + \frac{315}{128}\varrho^8\right);$$

$$I_{3,2}(\varrho) = -\frac{7\pi}{2048}\left(1 - \frac{25}{2}\varrho^2 + \frac{175}{4}\varrho^4 - \frac{525}{8}\varrho^6 + \frac{5775}{128}\varrho^8 - \frac{3003}{256}\varrho^{10}\right).$$

$$I_{0,-2}(\varrho) = -\frac{\pi}{2};\ I_{1,-2}(\varrho) = -\frac{9\pi}{8}\left(1 - \frac{5}{2}\varrho^2\right);$$

$$I_{2,-2}(\varrho) = -\frac{225\pi}{128}\left(1 - 7\varrho^2 + \frac{63}{8}\varrho^4\right);$$

$$I_{3,-2}(\varrho) = -\frac{1225\pi}{512}\left(1 - \frac{27}{2}\varrho^2 + \frac{297}{8}\varrho^4 - \frac{429}{16}\varrho^6\right).$$

Though the function $I_{n,m}(\varrho)$ is completely known, the physical problems mentioned at the beginning of the paper require the result in a different form. Equation (23) is remarkable insofar as the Legendre polynomial reappears in the result of integration; that is, the same function as occurred in the integrand of our integrals (1) and (14). Generally, the physical aspect of our problem requires that $I_{n,m}(\varrho)$ be written as a linear combination of the functions

$$F_{2n+1}(\varrho) = \frac{P_{2n+1}(\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}}. \quad \ldots \ldots \quad (33)$$

Therefore, let

$$I_{n,m}(\varrho) = \sum_{\nu=0}^{\infty} A_\nu(n,m)\, F_{2\nu+1}(\varrho), \quad \ldots \ldots \quad (34)$$

in which, of course, the series on the right actually has a finite number of terms. Using the orthogonality of the Legendre polynomials we have

$$A_\nu(n,m) = (4\nu+3)\int_0^{\pi/2} P_{2\nu+1}(\cos\vartheta)\, I_{n,m}(\sin\vartheta)\sin\vartheta\cos\vartheta\, d\vartheta =$$

$$= (4\nu+3)\, I_{n,m}(0)\int_0^{\pi/2} F(-n-\tfrac{1}{2}m-1,\, n-\tfrac{1}{2}m+\tfrac{1}{2};\, 1;\, \sin^2\vartheta)$$

$$P_{2\nu+1}(\cos\vartheta)\sin\vartheta\cos\vartheta\, d\vartheta =$$

$$= (4\nu+3) I_{n,m}(0)\sum_{\mu=0}^{\infty}\frac{\Gamma(n+\tfrac{1}{2}m+2)\,\Gamma(-n+\tfrac{1}{2}m+\tfrac{1}{2})}{\Gamma(n+\tfrac{1}{2}m+2-\mu)\,\Gamma(-n+\tfrac{1}{2}m+\tfrac{1}{2}-\mu)\,\Gamma(\mu+1)\,\Gamma(\mu+1)}$$

$$\int_0^{\pi/2} P_{2\nu+1}(\cos\vartheta)\sin^{2\mu+1}\vartheta\cos\vartheta\, d\vartheta =$$

$$= (4\nu+3)\, I_{n,m}(0)$$

$$\sum_{\mu=\nu}^{\infty}\frac{\Gamma(n+\tfrac{1}{2}m+2)\,\Gamma(-n+\tfrac{1}{2}m+\tfrac{1}{2})}{\Gamma(n+\tfrac{1}{2}m+2-\mu)\,\Gamma(-n+\tfrac{1}{2}m+\tfrac{1}{2}-\mu)\,\Gamma(\mu+1)\,\Gamma(\mu+1)}\, I_{\nu,2\mu+1}(0),$$

in which we have used eqs (15) and (27). Furthermore, on account of (16), we have

$$A_\nu(n,m) = (-1)^\nu \left(2\nu+\tfrac{3}{2}\right) I_{n,m}(0)$$

$$\sum_{\sigma=0}^{\infty} \frac{\Gamma(\nu+\frac{3}{2})\,\Gamma(n+\frac{1}{2}m+2)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+\frac{1}{2}m+2-\nu-\sigma)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2}-\nu-\sigma)\,\Gamma(2\nu+\frac{5}{2}+\sigma)\,\Gamma(\sigma+1)\,\Gamma(\nu+1)} =$$

$$= (-1)^\nu \left(2\nu+\tfrac{3}{2}\right) I_{n,m}(0)$$

$$\frac{\Gamma(\nu+\frac{3}{2})\,\Gamma(n+\frac{1}{2}m+2)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(2\nu+\frac{5}{2})\,\Gamma(n+\frac{1}{2}m+2-\nu)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2}-\nu)}$$

$$F\left(\nu-n-\tfrac{1}{2}m-1,\ \nu+n-\tfrac{1}{2}m+\tfrac{1}{2};\ 2\nu+\tfrac{5}{2};\ 1\right).$$

The remaining hypergeometric series can be summed; inserting moreover the known values of $I_{n,m}(0)$ we finally arrive at

$$A_\nu(n,m) = (-1)^{n+\nu}\left(\nu+\tfrac{3}{4}\right)$$

$$\frac{\Gamma(n+\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma(m+3)\,\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\nu+1)\,\Gamma(\frac{1}{2}m+2+n-\nu)\,\Gamma(\frac{1}{2}m+\frac{1}{2}-n-\nu)\,\Gamma(\frac{1}{2}m+\frac{7}{2}+n+\nu)\,\Gamma(\frac{1}{2}m+2-n+\nu)}. \qquad (35)$$

It is easily verified that the only values of ν for which $A_\nu(n,m)$ differs from zero are those satisfying $0 \leq \nu \leq n - \frac{1}{2}m + \frac{1}{2}$ (m odd) and $0 \leq \max\,(0, n-\frac{1}{2}m-1) \leq \nu \leq n + \frac{1}{2}m + 1$ (m even).

For the first few values of n and m one has

$$I_{0,-2}(\varrho) = -\frac{\pi}{2} F_1(\varrho); \qquad I_{1,-2}(\varrho) = -\frac{9\pi}{8} F_3(\varrho);$$

$$I_{2,-2}(\varrho) = -\frac{225\pi}{128} F_5(\varrho); \qquad I_{3,-2}(\varrho) = -\frac{1225\pi}{512} F_7(\varrho);$$

$$I_{0,0}(\varrho) = \frac{\pi}{5} F_1(\varrho) + \frac{\pi}{20} F_3(\varrho);$$

$$I_{1,0}(\varrho) = \frac{3\pi}{140} F_1(\varrho) + \frac{\pi}{20} F_3(\varrho) + \frac{5\pi}{224} F_5(\varrho);$$

$$I_{2,0}(\varrho) = \frac{5\pi}{352} F_3(\varrho) + \frac{25\pi}{832} F_5(\varrho) + \frac{525\pi}{36608} F_7(\varrho);$$

$$I_{0,1}(\varrho) = \tfrac{1}{3} F_1(\varrho); \qquad I_{1,1}(\varrho) = I_{2,1}(\varrho) = 0;$$

$$I_{0,2}(\varrho) = \frac{3\pi}{35} F_1(\varrho) - \frac{\pi}{45} F_3(\varrho) - \frac{\pi}{1008} F_5(\varrho);$$

$$I_{1,2}(\varrho) = -\frac{\pi}{105} F_1(\varrho) - \frac{\pi}{220} F_3(\varrho) - \frac{\pi}{728} F_5(\varrho) - \frac{5\pi}{27456} F_7(\varrho);$$

$$I_{0,3}(\varrho) = \frac{4}{15} F_1(\varrho) - \frac{2}{15} F_3(\varrho); \qquad I_{1,3}(\varrho) = -\frac{2}{35} F_1(\varrho);$$

$$I_{0,4}(\varrho) = \frac{2\pi}{21} F_1(\varrho) - \frac{3\pi}{44} F_3(\varrho) + \frac{3\pi}{728} F_5(\varrho) + \frac{\pi}{13728} F_7(\varrho);$$

$$I_{0,5}(\varrho) = \frac{64}{175} F_1(\varrho) - \frac{224}{675} F_3(\varrho) + \frac{8}{189} F_5(\varrho).$$

Eindhoven, July 1949.

Mathematics. — *Construction of a confidence region for a line.* By J. HEMELRIJK. (Communicated by Prof. D. VAN DANTZIG.)

(Communicated at the meeting of September 24, 1949.)

1. *Introduction.*

Let Γ be a *probability set* („Wahrscheinlichkeitsfeld" according to A. KOLMOGOROFF), i.e. a set Γ of elements λ, upon which an absolutely additive set function F is given (defined for all subsets Λ of Γ belonging to a given closed family H of subsets which contains Γ), with the properties

$$F(\Lambda) \geqq 0 \quad \text{for every} \quad \Lambda \in H$$

$$F(\Gamma) = 1.$$

Then a *random variable* $\underline{x}$ 1) can be considered as a function, defined for every $\lambda \in \Gamma$ and taking there the value $x(\lambda)$. If Λ is the subset of Γ, where $x(\lambda)$ takes a certain set X of values, the probability that $\underline{x} \in X$ (denoted by $P[\underline{x} \in X]$) is

$$P[\underline{x} \in X] = F(\Lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A *random element* $\underline{\varphi}$ of some set K (e.g. a random point or a random vector) can analogously be defined by adjoining an element φ of K to every element λ of a probability set Γ (notation: $\varphi(\lambda)$); and a *random system* $\underline{\Phi}$ *of elements* φ by adjoining a subset Φ of K to every element λ of a probability set Γ (notation: $\Phi(\lambda)$).

If $\underline{\Phi}$ is a random system of elements φ, and if φ_0 is one such element; if furthermore the random variable $\underline{u}(\varphi_0)$ is defined by the relations

$$u(\varphi_0;\lambda) = 1 \quad \text{if} \quad \varphi_0 \in \Phi(\lambda)$$

$$u(\varphi_0;\lambda) = 0 \quad \text{if not,}$$

then $\underline{\Phi}$ is called a *confidence region* for φ_0 with *confidence level* $p = P[\underline{u}(\varphi_0) = 0]$ (or: *confidence coefficient* $1-p = P[\underline{u}(\varphi_0) = 1]$).

2. *The problem.*

This may be formulated as follows:
Given: 1. a random set of n points $\underline{P}_i$ $(i = 1, \ldots, n)$ in a plane V satisfying the relations

$$\underline{P}_i = Q_i + \underline{\mathbf{w}}_i \qquad (i = 1, \ldots, n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

1) The random character of a variable, or, in general, of an element of some set, will be indicated by underlining the symbol, which denotes the variable or element respectively.

where $\underline{\mathbf{w}}_i$ is a random system of vectors in V [2]) and $Q_1, \ldots, Q_n$ are fixed points in V, situated on a straight line L, given by the equation

$$L \equiv \alpha\xi + \beta\eta + \gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ξ and η are Cartesian coordinates in V;

2. some conditions, which will be specified later, about the probability distribution of the random system of errors $\underline{\mathbf{w}}_i$;

3. a real number p with $0 < p < 1$;

To find: a confidence region $\underline{R}$ for L, consisting of lines in V and depending on $\underline{P}_1, \ldots, \underline{P}_n$ only, with confidence level $\leqq p$.

In particular we shall give a construction depending on $\underline{P}_1, \ldots, \underline{P}_n$ only, of confidence regions for:

I. The direction δ_0 of L $(\delta_0 = -\alpha/\beta)$

II. The intercept $\tau_0 = -\gamma/\beta$ of L, a) under the condition $\delta_0 = \delta$ and b) unconditionally

III. δ_0 and τ_0 jointly.

The construction of a joint confidence region for δ_0 and τ_0 is equivalent with the construction of $\underline{R}$. The constructions will be given in separate sections, the conditions concerning the probability distribution of the random set of errors $\underline{\mathbf{w}}_i$ being mentioned at the beginning of each section.

The probability set of the random system of errors $\underline{\mathbf{w}}_i$ $(i = 1, \ldots, n)$, for which we may take a $2n$-dimensional Cartesian space, will be called Γ. Each element $\lambda \in \Gamma$ then corresponds with a specified system of errors $\mathbf{w}_i(\lambda)$ and, Q_i being fixed, with a specified system of points $P_i(\lambda)$ $(i = 1, \ldots, n)$. Therefore Γ may be taken as the probability set of the random system of points $\underline{P}_i$ also.

3. *Remarks.*

3.1. The problem under consideration may arise in many fields of science, e.g. in physics, chemics and economics. If ξ and η are two variables, known (or supposed) to be linearly connected according to equation (3) with unknown coefficients α, β and γ; if furthermore the measurements of both ξ and η are subject to error; then the determination of a joint confidence-region for $-\alpha/\beta$ and $-\gamma/\beta$ by means of n observed points P_i with coordinates (x_i, y_i) $(i = 1, \ldots, n)$ is identical with our problem. To every observed point P_i a "true", but unknown, point Q_i is then supposed to correspond, according to the equations (2), where $\mathbf{w}_i$ represents the error of the ith observation. To these errors $\mathbf{w}_i$ corresponds an element $\lambda \in \Gamma$ (where Γ is the probability set of the errors, cf. 2.) and to this element λ corresponds a set $R(\lambda)$ of lines in V, which can be constructed by means of the points $P_1, \ldots, P_n$. $R(\lambda)$ may be regarded as an "observation" of the random confidence region $\underline{R}$ for L, corresponding to the observed points $P_1, \ldots, P_n$. The property, that the confidence coefficient is $1-p$ is then usually expressed by saying, that the probability, that L is an element of $R(\lambda)$ is equal to $1-p$.

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[2]) The vectors $\underline{\mathbf{w}}_i$ will be called the errors.

3. 2. A solution for a special case has been given by A. WALD in 1940 3). The conditions he imposes on the random set of errors are, however, rather more stringent (e.g. normality) than the conditions, which will be used here. In the same paper he derives consistent estimates of the coefficients $-\alpha/\beta$ and $-\gamma/\beta$ under less stringent conditions. We shall show that fairly general conditions are sufficient for the construction of confidence regions of these coefficients.

The methods used are different from those used by WALD, and of an elementary nature. They are related to those generally employed for the parameter-free construction of confidence intervals. The smallest number of points, which is needed for the construction of the confidence-regions mentioned above, with a reasonably large confidence coefficient (about 0,95) will prove to be seven.

Another partial solution of our problem, together with the solution of some other problems, has recently been found by H. THEIL. In particular he gives another confidence region for $-\alpha/\beta$ under conditions of the same nature as those imposed here, in a publication shortly to appear.

4. *Confidence region $\underline{D}$ for the direction δ_0 of L.*

4. 1. *Condition* I:

a. The n random errors $\underline{\mathbf{w}}_i$ $(i = 1, \ldots, n)$ are independently distributed with twodimensional probability distributions, which are the same for every i.

b. If $\underline{u}_i$ and $\underline{v}_i$ are the components of $\underline{\mathbf{w}}_i$ in the direction of the ξ- and η-axes of V, then the probability, that the random point with coordinates $\underline{u}_i$ and $\underline{v}_i$ lies on a fixed straight line N in V is equal to zero for every N in V (and for every i) 4).

Remark: strictly speaking it is sufficient if condition I *b* is fulfilled for all lines parallel to L only. In general however, L being unknown, this amounts to the same as I *b*.

4. 2. *Notation:* We shall call the strip (including its boundaries) of the plane V, bounded by two parallel straight lines through P_r and P_s $(r \neq s)$ and having the direction δ the $(r, s; \delta)$*-strip*. When $\underline{P}_r$ and $\underline{P}_s$ are random points, this strip is a random strip with fixed direction δ.

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3) The fitting of straight lines if both variables are subject to error, Ann. Math. Stat. 11 p. 284—300 (1940). This paper contains a summary of earlier results.

4) Interpreting a probability distribution as the distribution of a unit mass over the probability set, this means, that no straight line bears a positive mass.

A direction δ will be called $(r, s; m)$*-rejectable* with respect to the specified system of points P_i $(i = 1, \dots, n)$, if the $(r, s; \delta)$-strip corresponding to $P_1, \dots, P_n$, contains at least $n - m$ of the points $P_1, \dots, P_n$ and $(r, s; m)$-*acceptable* (where "acceptable" is short for "non-rejectable") if this is not the case.

The absolutely additive set function F on the probability set Γ (cf. 2) then determines the probability, that a fixed direction δ will be $(r, s; m)$-rejectable for given r and s: if $\Lambda \subset \Gamma$ is the subset of those λ, for which δ is $(r, s; m)$-rejectable, then $F(\Lambda)$ is this probability.

4. 3. *Theorem* I:

If $\underline{P}_i = Q_i + \underline{w}_i$ $(i = 1, \dots, n)$ *are n random points in a plane V, where* $Q_i, \dots, Q_n$ *lie on a straight line L in V and* $\underline{w}_1, \dots, \underline{w}_n$ *fulfill condition I, then for any fixed r and s* (*with* $r \neq s$; $1 \leqq r \leqq n$; $1 \leqq s \leqq n$) *and for any natural number* m $(0 \leqq m \leqq n - 2)$, *the set* $\underline{D}$ *of* $(r, s; m)$*-acceptable directions is a confidence region for the direction* δ_0 *of L, with confidence level*

$$p_1 = \frac{(m+1)(m+2)}{n(n-1)} \qquad (0 \leqq m \leqq n-2). \quad . \quad . \quad . \quad . \quad (4)$$

Proof: To prove the theorem, we only have to show that the probability, that δ_0 is $(r, s; m)$-rejectable is equal to p_1.

Now δ_0 is $(r, s; m)$-rejectable, if and only if the $(r, s; \delta_0)$-strip contains at least $n - m$ of the points $\underline{P}_1, \dots, \underline{P}_n$. Denoting the distance from $\underline{P}_i$ to L, measured in an arbitrary fixed direction different from δ_0, by $\underline{z}_i$, this means, that at least $n - m$ of the quantities $\underline{z}_1, \dots, \underline{z}_n$ lie in the closed interval $(\underline{z}_r, \underline{z}_s)$. According to condition I the $\underline{z}_i$ $(i = 1, \dots, n)$ are distributed independently, according to a probability distribution, which is the same for every i and which is continuous because of condition I *b*. Therefore the probability is equal to one, that all $\underline{z}_i$ are different and, arranging them according to decreasing magnitude, $\underline{z}_r$ has, for every j, probability $1/n$ to be the jth one from the top. If δ_0 is $(r, s; m)$-rejectable, $\underline{z}_r$ must have one of the $m + 1$ largest or one of the $m + 1$ smallest values and if it takes the jth value (with $j \leqq m + 1$) from the top (or from the bottom respectively), then $\underline{z}_s$ must take one of the $m + 2 - j$ smallest (or largest) values respectively. The probability, that δ_0 is $(r, s; m)$-rejectable is therefore equal to

$$2 \sum_{j=1}^{m+1} \frac{1}{n} \cdot \frac{m+2-j}{n-1} = \frac{(m+1)(m+2)}{n(n-1)}$$

which is equal to p_1.

Remark: $\underline{D}$ consists of a finite number of angles [5]) corresponding with a finite number of intervals for $-\alpha/\beta$. It reduces to one angle if there is a $(r, s; \delta)$-strip, which contains all points $P_1, \ldots, P_n$. This condition is sufficient, but not necessary.

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[5]) Where "angle" stands for "pair of vertically opposite angles".

4.4. *On the choice of the numbers r and s.*

Theorem I has been proved without imposing any restrictions on the choice of $\underline{P}_r$ and $\underline{P}_s$ out of $\underline{P}_1, \ldots, \underline{P}_n$. It must, however, be pointed out, that this choice must be independent of the vectors $\underline{\mathbf{w}}_i$, because otherwise, the $\underline{z}_i$ $(i = 1, \ldots, n)$ do not necessarily have the same probability distribution any more.

Bearing this restriction in mind, we now consider the question, which choice would, on the average, be preferable. It is clear that, unless the points $\underline{P}_i$ lie on a straight line (in which case every direction is $(r, s; m)$-rejectable; this case, however, has probability zero), the direction $\underline{d}_{rs}$ of the line $\underline{P}_r\underline{P}_s$ is always $(r, s; m)$-acceptable. Clearly, the method will gain in power, if we do not choose r and s arbitrarily, but if we choose them so, that the probability of a large deviation of $\underline{d}_{rs}$ from the direction δ_0 of L, is minimised. A choice of r and s, which attains this end for every deviation, will therefore be considered preferable.

Supposing the indices of the points Q_i $(i = 1, \ldots, n)$ to be chosen such, that $Q_1 \neq Q_n$ and that Q_i for $i = 2, \ldots, n-1$ lies in the open interval (Q_1, Q_n), it is easy to see, that the choice $r = 1$ and $s = n$ (or $s = 1$, $r = n$) is preferable in the abovementioned sense to all other choices.

To prove this, we consider, for every r and s with $r \neq s$, the two-dimensional probability set N_{rs} of the random system of the two vectors $\underline{\mathbf{w}}_r$ and $\underline{\mathbf{w}}_s$. N_{rs}, as well as the absolutely additive set function on N_{rs} representing the joint probability distribution of $\underline{\mathbf{w}}_r$ and $\underline{\mathbf{w}}_s$, are the same for every r and s $(r \neq s)$. Every element μ of N_{rs} corresponds with a deviation $\triangle_{rs}(\mu)$ of $d_{rs}(\mu)$ from δ_0. This deviation $\triangle_{rs}(\mu)$, however, is, for every element μ of N_{rs}, smallest if $r = 1$ and $s = n$ (or $r = n$ and $s = 1$), which follows easily from the fact, that Q_1 and Q_n are the extreme point of $Q_1, \ldots, Q_n$. This proves our contention.

A second reason, for preferring the choice $r = 1$ and $s = n$ is, that, according to the remark of the preceding section, the probability that $\underline{D}$ consists of a single angle, is then as large as possible.

In general it will not be possible to select from a specified system of points P_i $(i = 1, \ldots, n)$ the points P_1 and P_n corresponding to Q_1 and Q_n, without making any further assumptions about the errors, because the points Q_i are unknown. It may occur, that the points P_1 and P_n can be selected on non-statistical considerations, for instance if it is known, that the points $Q_1, \ldots, Q_n$ form a monotonous sequence, being observed in the same order (e.g. if ξ denotes the time when the observation takes place). Another situation, which may arise is, that we have a criterion C at our disposal, which (under some further assumptions for the errors) indicates unambiguously among every specified system of points $P_1, \ldots, P_n$ (except perhaps with zero probability) two points P_r and P_s with $r \neq s$ and with the property, that

$$P[(r=1 \text{ and } s=n) \text{ or } (r=n \text{ and } s=1)] \geqq 1-q. \quad . \quad . \quad (5)$$

C may, for instance, consist of taking the point P_i with smallest abscissa as P_r and the one with largest abscissa as P_s. We shall not occupy ourselves with a discussion of the different possibilities for C and the computation of the corresponding q, this being quite a subject in itself [6]. We only point out, that, if (5) is valid, theorem I remains correct with confidence level $p_1^* \leq p_1 + q - p_1 q$, if for $\underline{P}_r$ and $\underline{P}_s$ we take the points indicated by C, instead of keeping r and s constant. This may be seen as follows: denoting by A the event, that C has indicated the right pair of points, we have

$$P[\delta_0 \in D \mid A] = 1 - p_1$$

i.e. the conditional probability, under the condition A, that $\delta_0 \in D$, is equal to $1 - p_1$. Thus:

$$P[\delta_0 \in D] \geqq P[A \text{ and } \delta_0 \in D] = P[A] \cdot P[\delta_0 \in D \mid A] \geqq (1-q)(1-p_1)$$
$$p_1^* = 1 - P[\delta_0 \in D] \leqq 1 - (1-q)(1-p_1) = p_1 + q - p_1 q.$$

5. *Conditional confidence interval* $\underline{T}$ *for* $\tau_0 = -\gamma/\beta$ *under the condition* $\delta_0 = \delta$.

5.1. *Condition* II: The random vectors $\underline{\mathbf{w}}_i$ $(i = 1, \ldots, n)$ are distributed independently; for every i the distribution of $\underline{\mathbf{w}}_i$ is such, that the random point $\underline{P}_i$ has equal probability to lie on either side of L and probability zero to lie on L.

Remark: The distribution of $\underline{\mathbf{w}}_i$ may now depend on i. Condition II is satisfied if e.g. the distribution of $\underline{\mathbf{w}}_i$ (for every i) is symmetrical with respect to the origin.

5.2. *Notation:* We shall call τ a (δ, k)*-rejectable* value of the intercept, under the condition $\delta_0 = \delta$, with respect to a specified system of points $P_1, \ldots, P_n$, if at most k of the points P_i $(i = 1, \ldots, n)$ are situated on one side of the line L' through the point $(0, \tau)$ with direction δ.

If on both sides of L' lie more than k points, τ will be called (δ, k)*-acceptable* under the condition $\delta_0 = \delta$ (where again "acceptable" is short for "non-rejectable"). The condition $\delta_0 = \delta$ will not always be mentioned explicitly.

The absolutely additive set function F on Γ then determines the probability, that a fixed value τ will be (δ, k)-rejectable: if $\Lambda \subset \Gamma$ is the subset of those λ, for which τ is (δ, k)-rejectable, then $F(\Lambda)$ is this probability.

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[6] It is clear that especially if the length of $\underline{\mathbf{w}}_i$ has a finite range, q will be equal to 0, if the distance of the points with smallest and largest abscissae (or ordinates) have a distance larger than *four* times this range to all other points.

5.3. *Theorem* II: *If* $\underline{P}_i = Q_i + \underline{\mathbf{w}}_i$ $(i = 1, \ldots, n)$ *are n random points in a plane* V, *where* $Q_1, \ldots, Q_n$ *lie on a straight line* L *in* V *and* $\underline{\mathbf{w}}_1, \ldots, \underline{\mathbf{w}}_n$ *fulfill condition II, then the set* $\underline{T}$ *of* (δ, k)*-acceptable values* τ *(where* k *is an integer* $< \frac{n-3}{2}$*) of the intercept is a conditional confidence interval for the intercept* τ_0 *under the condition* $\delta_0 = \delta$, *with confidence level*

$$p_2 = 2^{-n+1} \sum_{i=0}^{k} \binom{n}{i} \qquad \leqq k < \tfrac{n-3}{2}) \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Proof: From the definition of a (δ, k)-rejectable value τ it is clear, that the set $\underline{T}$ is a random interval. Remains to calculate the confidence level. For every point $\underline{P}_i$ the probability to lie at either side of L is equal to $\frac{1}{2}$. Therefore the probability, that at one of the sides of L lie k or less points $\underline{P}_i$, is equal to

$$2^{-n+1} \sum_{i=0}^{k} \binom{n}{i} .$$

6. *Confidence region* $\underline{R}$ *for* L.

6.1. *Condition* III: conditions I and II are both satisfied; i.e.: a. The errors $\underline{\mathbf{w}}_i$ are independently distributed, with two dimensional probability distributions, which are the same for every i.

b. The probability, that $(\underline{u}_i, \underline{v}_i)$, where $\underline{u}_i$ and $\underline{v}_i$ are the components of $\underline{\mathbf{w}}_i$, lies on a fixed straight line parallel to L, is equal to zero, for every such line.

c. The probability, that $\underline{P}_i$ lies above L is equal to $\frac{1}{2}$. The probability, that $\underline{P}_i$ lies on L is equal to zero.

6.2. *Theorem* III: *If* $\underline{P}_i = Q_i + \underline{\mathbf{w}}_i$ $(i = 1, \ldots, n)$ *are n random points in a plane* V, *where* $Q_1, \ldots, Q_n$ *lie on a straight line* L *in* V *and* $\underline{\mathbf{w}}_1, \ldots, \underline{\mathbf{w}}_n$ *fulfill condition III; if r and s are two different integers taken from* $1, \ldots, n$*; if m is an integer with* $0 \leqq m \leqq n-2$ *and if k is an integer with* $0 \leqq k < \frac{n-3}{2}$*; then the set* $\underline{R}$ *consisting of those lines in V of which both the direction* δ *is* $(r, s; m)$*-acceptable and the intercept* τ *is* (δ, k)*-acceptable, is a confidence region for L with confidence level*

$$p = p_1 + p_2 - p_1 p_2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

where

$$p_1 = \frac{(m+1)(m+2)}{n(n-1)} \quad \text{and} \quad p_2 = 2^{-n+1} \sum_{i=0}^{k} \binom{n}{i} .$$

Proof: The proof consists again of showing, that the set $\underline{R}$ has probability p not to contain L. According to theorem I and II, the probabilities, that δ_0 is $(r, s; m)$-rejectable and that τ_0 is (δ_0, k)-rejectable are respectively p_1 and p_2. Now the $(r, s; m)$-rejectability of δ_0 depends on the place which $\underline{z}_r$ and $\underline{z}_s$ (cf. the proof of theorem I) take in the sequence of $\underline{z}_1, \ldots, \underline{z}_n$ when arranged according to decreasing magnitude. The (δ_0, k)-rejectability of τ_0, however, is invariant against permutations of the points $\underline{P}_i$; hence the $(r, s; m)$-rejectability of δ_0 and the (δ_0, k)-rejectability of τ_0 are independent. From this (7) follows.

6.3. *The actual construction of* $\underline{R}$.

In diagram 1 an example is given of the form which the set $\underline{R}$ can take in a specified case (i.e. for one element λ of Γ). P_r and P_s have been supposed to be the points with smallest and largest abscissa (cf. 4.4; if e.g. the error in the ξ-direction is sufficiently small in comparison with the differences of the abscissae of these points and the other points, this procedure is justified).

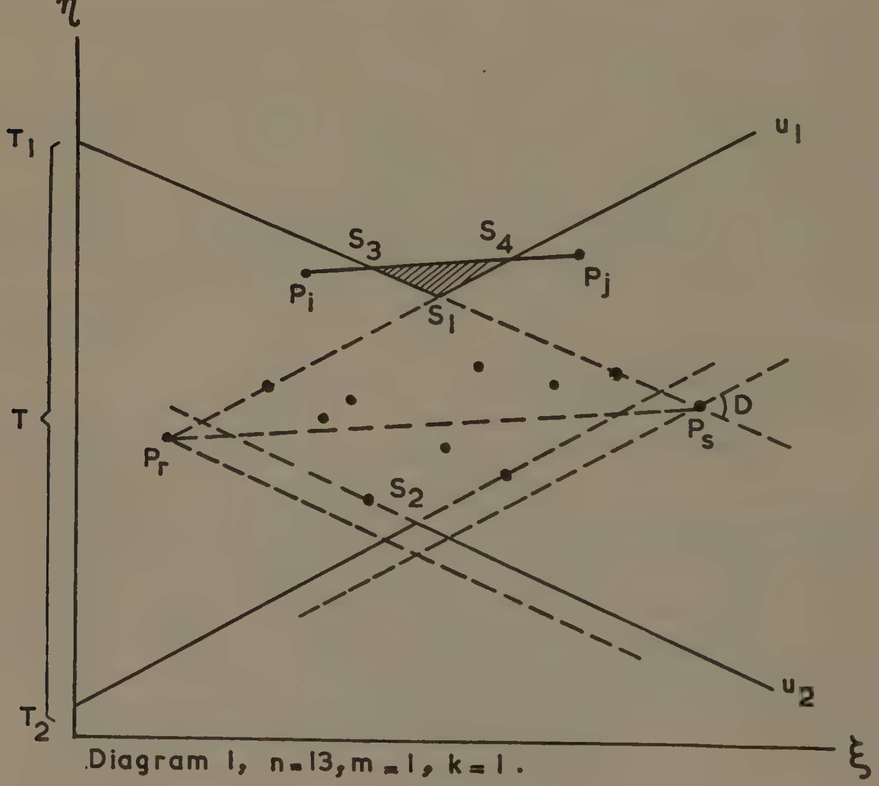

Diagram 1, $n = 13, m = 1, k = 1$.

First D is constructed by letting two parallel lines revolve around P_r and P_s respectively and registering the $(r, s; m)$-acceptable directions. Then the parallel lines through P_r and P_s in both extreme acceptable directions are pushed together (or eventually pulled apart) untill a position is reached, where they indicate the extreme lines of their own direction δ which are (δ, k)-acceptable. This gives the "diabolo" $T_1 S_1 U_1$; $T_2 S_2 U_2$.

Next all points P_i lying outside one of the strips bounded by $T_1 S_1$ and $S_2 U_2$ or $T_2 S_2$ and $S_1 U_1$ respectively are connected by straight lines. From these those lines are selected, which have an $(r, s; m)$-acceptable direction δ, and which have an intercept τ which is (δ, k)-acceptable or on the verge of (δ, k)-acceptability (like $P_i P_j$ in the diagram). The portions (like $S_3 S_1 S_4$) which these lines cut of from $T_1 S_1 U_1$ and $T_2 S_2 U_2$ are joined to the diabolo.

The resulting region of the plane V then contains all lines of R; a line however, lying in this region, does not necessarily belong to R because, although its direction δ is acceptable, its intercept may be (δ, k)-rejectable.

In the diagram we have $n = 13$, $m = k = 1$; hence $p = 0.043$.

The construction can easily be carried out graphically by taking e.g. a very large η-scale, so that the ordinates of the points P_i have a large variation.

7. *Miscellanous remarks.*

7.1. *Unconditional confidence interval for τ_0.*

The set of those points of the η-axis, which lie on a line of $\underline{R}$, is a confidence interval for τ_0 with confidence level p, without condition about the direction of L. In diagram 1 this interval is (T_1, T_2).

7.2. *Conditional confidence region for δ_0 under the condition $\tau_0 = \tau$.*

This confidence region consists of the direction of those lines through the point $(0, \tau)$, for which:

1. The direction δ is $(r, s; m)$-acceptable.
2. τ is (δ, k)-acceptable.

The confidence level then is p.

The set of directions δ of those lines through the point $(0, \tau)$, for which τ is (δ, k)-acceptable, is another conditional confidence region for δ_0, containing the first one, with confidence level p_2.

7.3. *Testing of hypotheses.*

From the foregoing sections simple tests can be derived for the hypotheses, *a*) that L is a given line L', and *b*) that L contains a given point Q_0.

The test of the hypothesis $L' = L$ consists of drawing two lines L_1' and L_2' parallel to L' through P_r and P_s and counting the number of points P_i outside the strip bounded by L_1' and L_2'. Calling this number m' and calling the numbers of points P_i lying on the two sides of L', k' and k'' respectively, the hypothesis $L' = L$ is rejected if either $m' \leqq m$ or $\text{Min}\,(k', k'') \leqq k$ (i.e. if L' does not belong to R). The level of significance of this test is $p = p_1 + p_2 - p_1 p_2$.

In an analogous way the other hypothesis mentioned may be tested without carrying out the complete construction of R.

7.4. *Table of p_1 and p_2.*

n ↓ \ m, k →	0	1	2	3	4	5
6	0.067 0.032					
7	0.048 0.016					
8	0.036 0.008	0.108 0.071				
9	0.028 0.004	0.084 0.040				
10	0.022 0.002	0.067 0.022				
11	0.018 0.001	0.055 0.012	0.110 0.066			
12	0.015 0.0005	0.046 0.007	0.091 0.039			
13	0.013 0.0003	0.039 0.004	0.077 0.023	 0.093		
14	0.011 0.0002	0.033 0.002	0.066 0.013	 0.058		
15	0.010 0.00006	0.029 0.001	0.058 0.008	0.096 0.036		
16	0.009 0.00004	0.025 0.0006	0.050 0.005	0.084 0.022	 0.077	
17	0.008 0.00002	0.023 0.0003	0.045 0.003	0.074 0.013	 0.050	
18	0.007 0.000008	0.020 0.0002	0.040 0.002	0.066 0.008	0.099 0.031	 0.097
19	0.006 0.000004	0.018 0.00008	0.036 0.0008	0.059 0.005	0.088 0.020	 0.064
20	0.006 0.000002	0.016 0.00005	0.032 0.0005	0.053 0.003	0.079 0.012	 0.042

$$p_1 = \frac{(m+1)(m+1)}{n(n-1)} \qquad p_2 = 2^{-n+1} \sum_{i=0}^{k} \binom{n}{i}.$$

In every compartment the number at the top represents p_1 and the number at the bottom p_2; p_1 and p_2 need not be taken from the same partition in a row.

Values of p_1 have been included up to about 0.10 and of p_2 to such a level, that with the same n there is a p_1 which makes $p_1 + p_2$ not larger than about 0.10; the reason for including these rather high values is, that in special cases regions may be indicated corresponding with a confidence level of $\frac{1}{2}(p_1 + p_2) - \frac{1}{4} p_1 p_2$. In the diagram of 6. 3 e.g. the part of the η-axis above T_1 contains τ_0 with this probability only, if the error in the ξ-direction is so small, that the abscissa of P_r must necessarily be smaller than the abscissa of P_s for every $\lambda \in \Gamma$, and that, at the same time, no point P_i has a negative abscissa for any $\lambda \in \Gamma$. The same property then holds for the part of the η-axis below T_2. We omit the proof of this contention; it runs along the same lines as the proofs of the other theorems, applying one-sided criteria for rejectability instead of the two-sided criteria, which have been used there.

I want to thank Prof. Dr D. VAN DANTZIG, whose suggestions helped me to give the paper its final form.

Publication of the Statistical Department of the "Mathematisch Centrum", Amsterdam.

Chemistry. — *A study on the refraction of coacervates.* By H. L. BOOIJ, J. C. LYCKLAMA *) and C. J. VOGELSANG *) [1]. (Communicated by Prof. H. G. BUNGENBERG DE JONG.)

(Communicated at the meeting of October 29, 1949.)

1. *Introduction.*

The historical development of the investigations on coacervation phenomena had the logical consequence that in many experiments the volume of the coacervate layer was used as a criterion for the relations between the interacting colloids and ions. The general idea was that a strong interaction is combined with an expulsion of water and that in this way the volume of the coacervate layer decreases. It was observed, however, that the reverse of this rule may not always be true (for instance in the gelatin-gum arabic coacervate the largest coacervate volume is observed when the interaction between the colloids is strongest). In these cases additional experiments must be performed to analyse the relations between the colloids. Generally speaking, one must then make an analysis of the coacervate layer as well as of the equilibrium fluid (BUNGENBERG DE JONG and DEKKER). The method which we describe here may in some cases make these tedious analyses superfluous.

2. *Method.*

The idea which forms the basis of our method is very simple: the difference between the coacervate layer and the equilibrium liquid is that most of the colloid is found in the coacervate, which must result in a difference between the refractive indices of the two layers. When the concentration of the colloids is high, we must expect that this will be reflected in the refractive index of the coacervate layer. This we tried to verify with the aid of two types of refractometers: the "Eintauch Refraktometer" (ZEISS) and the "Abbe refractometer" (HILGER).

First of all we examined the coacervation of an oleate solution under the influence of high concentrations of KCl (BOOIJ and BUNGENBERG DE JONG). At 1.5 mol/l KCl a coacervate appears, the volume of which decreases rapidly with increasing concentration of KCl (fig. 1 *A*). We measure the volume of the coacervate layers in % of the total volume and plot these numbers against the KCl concentration (fig. 1 *B*). Finally fig. 1 *C* shows the refraction of the soap solution before coacervation (*S*), that of the coacervate layers (*C*) and the equilibrium liquids (*E*). It is evident that the refraction of the coacervate layer rises steeply as the concentration of

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[1] Publication no. 1 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

KCl is increased (and thus the volume of the layer decreases). The slope of line S is, of course, the result of the rising concentration of KCl and it is of fundamental importance that line E runs parallel to S. E starts (at the point of coacervation) at a lower value of refraction, which means that soap has disappeared from the equilibrium liquid, and is concentrated in the coacervate layer.

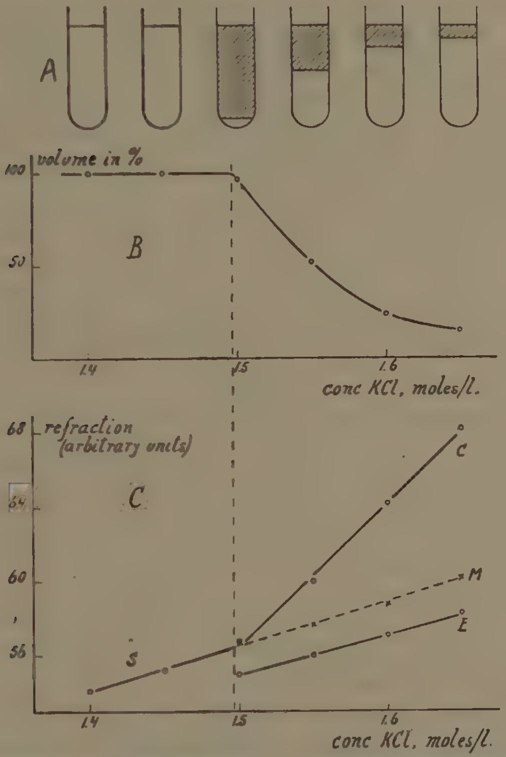


Fig. 1. *A*. Coacervation of a Na-oleate solution under the influence of added KCl. — *B*. The volume of the coacervate layers of the test tubes of fig. 1 *A* as a function of the KCl concentration. — *C*. Refraction indices of *a*) S = uncoacervated solutions, *b*) C = coacervate layers and *c*) E = equilibrium liquids of the test tubes of fig. 1 *A*. M = "mean refraction", computed from coacervate volume and C and E.

Line E can only be parallel to S if two conditions are fulfilled.

1) the colloid is concentrated practically totally in the coacervate layer (or is present in a constant amount in the equilibrium fluid, which would be very improbable in a medium with increasing salt concentration).

2) the KCl concentration in the equilibrium and the coacervate layer is essentially the same (this could be easily demonstrated by making a "synthetic" equilibrium fluid and comparing its refraction with that of the real equilibrium liquid).

In this experiment both conditions are fulfilled and when in another type of coacervation a different diagram results, one can find pretty easily which condition does not hold. In general we expect that aberrations from factor 1 will be found to be quite common, especially with colloids which are very soluble in water. It does not seem probable that an unequal distribution of the salt will ever play a great part in coacervation phenomena (except at low concentrations).

A combination of the data of coacervate volumes and refraction shows us another useful rule. Suppose that in a certain case the volume of the coacervate layer is V_c (with a refraction n_c), while the volume of the equilibrium liquid is V_e (with a refraction n_e). Then we might compute a "mean refraction" n_m, which would be the refraction of the system if this had not been coacervated.

$$\frac{n_c \times V_c + n_e \times V_e}{V_c + V_e} = n_m \quad \dots\dots\dots \quad (1)$$

In the case of fig. 1 we find the following values for n_m (table I).

TABLE I.

V_c	n_c	V_e	n_e	n_m
97	56.8	3	55.0	56.75
42	60.0	58	56.0	57.7
24	64.2	76	57.1	58.8
15	68.2	85	58.3	60.2

The values of n_m have been put into fig. 1 (M) and it is clear that they lie on line S. This gives us yet another possibility for the investigation of coacervates; one might compute along this way the volume of very small coacervate layers (too small to measure along the direct way). (See following section).

The influence of organic substances on soap coacervates has been measured in two ways. The first was as follows (BUNGENBERG DE JONG, BOOIJ and SAUBERT): in a series of test tubes the amount of KCl is kept constant, so that in the blank a certain amount of coacervate is formed. To these tubes we add ever more of the substance to be investigated. Under the influence of this substance we see the coacervate layer increasing (for ethylalcohol see fig. 2 A and B) or decreasing (for butylalcohol see fig. 3 A and B). We may ask ourselves whether this increase of volume (fig. 2 B) means that the coacervate takes up ever more water. Measurements (fig. 2 C) of the refraction of coacervate layer (C) and equilibrium liquid (E) show that this is the case indeed, the coacervate being ever more diluted, till it disappears completely. In the case of butylalcohol (fig. 3 C) the reverse is true, the colloid being more and more concentrated as the concentration of butylalcohol increases. Here there is no indication of soap going into the equilibrium liquid.

The second method (BOOIJ and BUNGENBERG DE JONG) consisted in keeping the concentration of added organic substance constant and changing the KCl concentration.

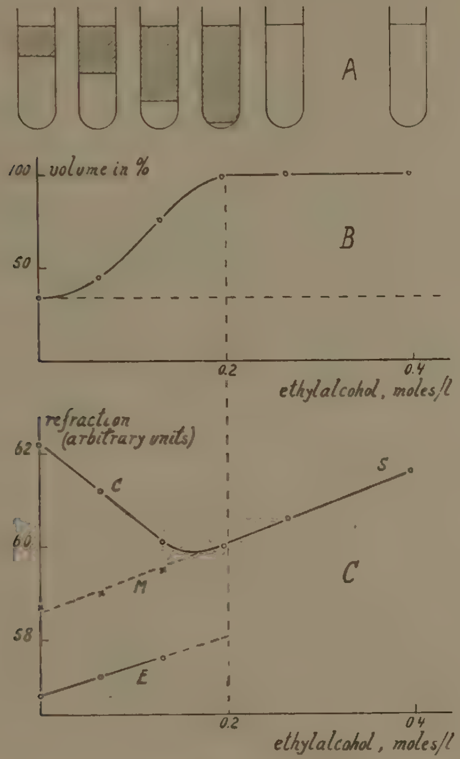
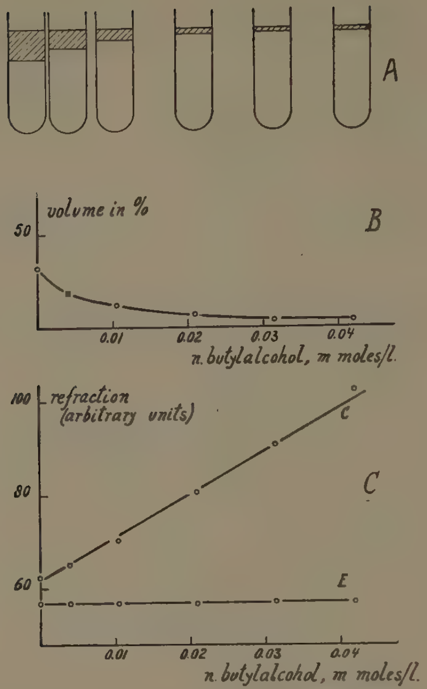


Fig. 2. *A*. Disappearance of an oleate coacervate (KCl constant) under the influence of ethylalcohol. — *B*. Volume of coacervate layers. — *C*. Refraction indices of *a*) *C* = coacervate layers, *b*) *E* = equilibrium liquids and *c*) *S* = uncoacervated solutions (*M* = "mean refraction").

3. *Experiments on very small coacervate layers.*

The study of phosphatide-coacervates (so important for many biological problems) has been hampered by two difficulties: 1) it is not easy to obtain reliable and stabile material in adequate quantity and 2) the coacervate layers are very small, so that the measurements of the volumes show grave errors.

We investigated whether our method would be valuable for the study of small and concentrated coacervate layers. Fig. 4 *A* shows the influence of *n* butylalcohol on oleate coacervate. It is obvious that coacervate layers of 10 % and less cannot be measured accurately. The refraction indices, however, can be measured very well (fig. 4 *B*) and it appears that the lines *C* for the blank and for *n* butylalcohol 30.5 m mol/1 are practically parallel. The shift in KCl concentration is comparable too in this case. So it seems justified to use the method of refraction measurements when dealing with small coacervate volumes.



Fif. 3. *A*. Under the influence of *n* butylalcohol the oleate coacervate is more condensed. — *B*. Volume of the coacervate layers. — *C*. Refraction indices of *a*) *C* = coacervate layers and *b*) *E* = equilibrium liquids.

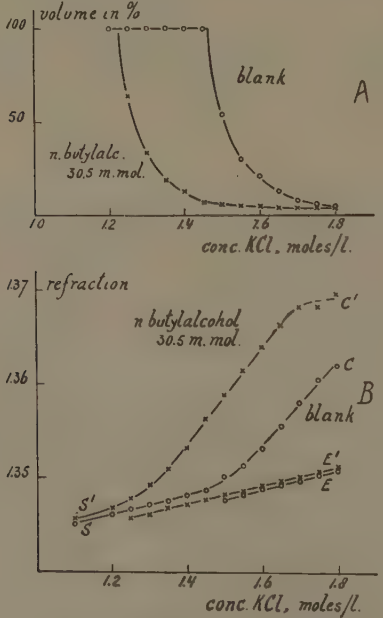


Fig. 4. *A*. If we add a constant quantity of *n* butylalcohol to an oleate solution, the KCl curve shifts to the left. — *B*. This can also be seen from the refraction indices (*C* = coacervate layer and *E* = equilibrium liquid of the blank; *C'* = coacervate layer and *E'* = equilibrium liquid of the experiments with *n* butyalcohol).

4. *On the determination of the coacervate volume from the "mean refraction"* n_m.

We have seen that we can compute a "mean refraction" (n_m) from the refraction indices and the volumes of a coacervated system. The values of n_m lie exactly on the line *S*, combining the refractive indices of the uncoacervated systems. This has been proved true for all coacervates investigated.

In reverse it must be possible to determine the volumes of the coacervate layers from refraction data only. This is demonstrated in fig. 5. The line for the "mean refractions" is constructed by producing the line *S*.

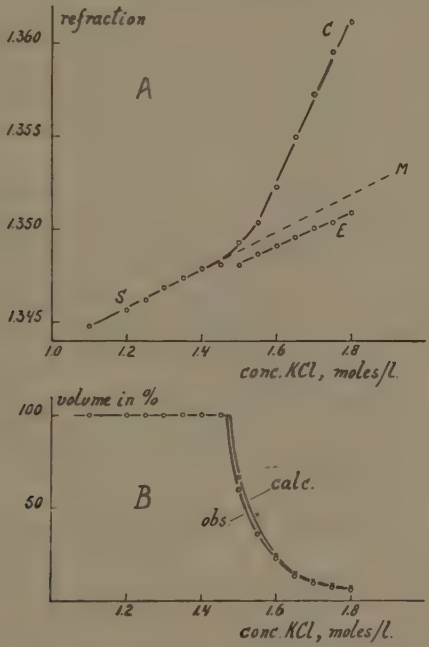


Fig. 5. From refraction data (*A*) one can calculate — with the aid of the "mean refraction" *M* — the coacervate volumes. *B*. Calc. = calculated values, obs. = direct measurements.

We have found already:

$$\frac{n_c\,V_c + n_e\,V_e}{V_c + V_e} = n_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When we express the volumes of the layers in per cent., we get:

$$V_c + V_e = 100$$
$$V_e = 100 - V_c .$$

When introduced into (1) this will give:

$$\frac{V_c\,(n_c - n_e)}{100} + n_e = n_m \qquad V_c = 100\,\frac{n_m - n_e}{n_c - n_e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this equation $(n_m - n_e)$ is a constant, i.e. the vertical distance between the parallel lines S and E. In table II we assembled the computed coacervate volumes $(\triangle n = (n_c - n_e) \times 10^4)$.

TABLE II.

Coacervate volumes computed from refraction data

$(n_m - n_e = 0{,}0008 = 8 \times 10^{-4})$

Conc. KCl	n_c	n_e	$\triangle n$	$V_{c\,(\text{calc.})}$	$V_{c\,(\text{obs.})}$
1.50	1.3493	1.3481	12	67	60
1.55	1.3504	1.3487	17	47	36
1.60	1.3523	1.3491	32	25	23
1.65	1.3550	1.3496	54	15	14
1.70	1.3573	1.3501	72	11	10
1.75	1.3596	1.3504	92	9	8
1.80	1.3612	1.3509	103	8	6

The largest discrepancies between $V_{(\text{calc.})}$ and $V_{(\text{obs.})}$ are found in the region of the large volumes of coacervate. This would be expected as here $(n_c - n_e)$ has the lowest value, and consequently the error will be large. At small coacervate volumes the error is much less, but it must be realised that the values computed depend strongly on $(n_m - n_e)$, and that this value must be determined as accurately as possible. Concluding we might say that the method of refraction measurement gives the best results at concentrations where the method of the direct volume measurements begins to fail.

5. *The region of coacervation in the phase diagram of soap.*

From the phase diagram of K-oleate-KCl-H_2O (Mc Bain) one can read off that coacervation may occur in two ways. The phenomenon pictured in fig. 1 results when the concentration of soap is relatively high. This can be easily demonstrated by a study of the coacervation region of the phase diagram (fig. 6). When the KCl concentration of an oleate solution is increased coacervation will take place eventually, if we proceed towards point c (fig. 6). The amounts ¹) of coacervate (V_c) and equilibrium

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¹) As the specific weights of coacervate layer and equilibrium liquid do not differ very much, it is justified as a first approximation to speak of volume instead of weight of the layers.

As regards fig. 6 it is important to observe that one changes from coacervates to equilibrium liquids when following the curve downwards. Then there must be a critical point (situated on the left bend of the curve) where it is not possible to decide whether one may speak of coacervate or equilibrium liquid.

While the general shape of the curve is practically not changed by impurities, it is certain that — quantitatively speaking — the curve will be found at other concentrations of oleate or KCl. Technical preparations of Na-oleate may contain large quantities of impurities, especially other soaps.

liquid (V_e) may be found from the phase diagram — they are in the proportion as the lines *bq* and *bp*:

$$\frac{V_c}{V_e} = \frac{bq}{bp}.$$

This means that in point *a* (coacervation just appearing) practically all of the liquid will be coacervate, while the equilibrium liquid is very small.

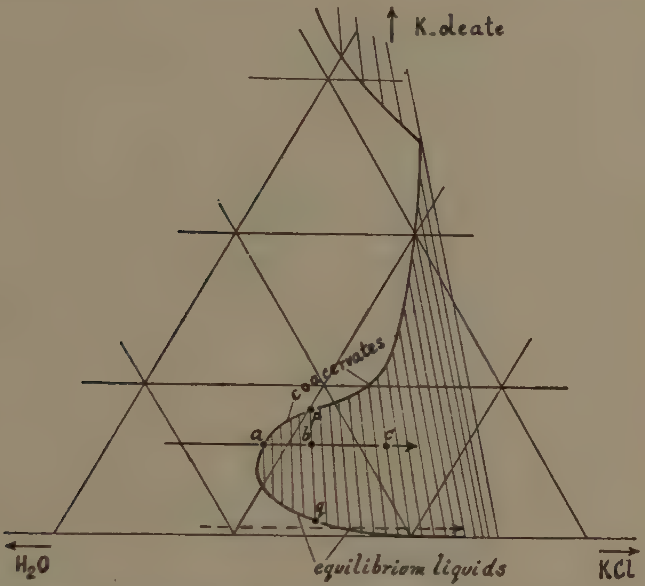

Fig. 6. Part of the phase-triangle oleate — KCl — H_2O (according to McBain).

All examples quoted thus far (figs. 1, 4 and 5) belong to this type of coacervation.

In lower concentrations of oleate another type of coacervation will be possible (fig. 6, dotted line). It is easily seen that here — at the beginning of coacervation — the reverse takes place. At first the coacervate layer will grow from zero to a certain value, after which the normal decrease of coacervate volume with increasing KCl concentration takes place.

The conclusion of these experiments is that it is possible to judge from some experiments on the refraction of coacervate layers where the demixing figures of soaps lie approximately. The phase diagram of soaps is obviously much influenced by the nature of the carbon chain. Soaps which are very soluble in water will give the diagram of fig. 6 at higher concentrations.

We made some experiments on a more or less purified solution of sulforicinolate (technical sulforicinolate contains much sodiumsulfate, which we removed by acidifying and taking up the sulforicinolic acid in ether).

The coacervate appears in small volume by adding KCl to a highly concentrated solution of "purified" sulforicinolate (fig. 7 *A*). It is obvious that we are dealing here with the lower half of a demixing figure (see fig. 6, dotted line). This is also clearly visible in fig. 7 *B* where the refraction data are plotted. The difference with fig. 1 *C* is that in experiments with oleate the first coacervate layer resembles the original solution very much, while the equilibrium liquid changes abruptly. With sulforicinolate the coacervate layer differs from the original solution (fig. 7 *B*), whereas the equilibrium liquid shows only a small change. This must also mean that in this case the colloid is present in the equilibrium liquid in fairly high amounts.

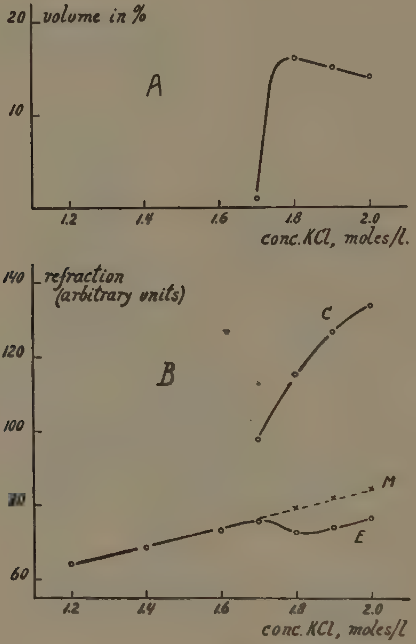

Fig. 7. *A*. Coacervate volumes of a "purified" sulforicinolate in relation to KCl concentration. — *B*. From refraction data it appears that at beginning coacervation much soap is present in the equilibrium liquid (*E*).

A crude solution of sulforicinolate also coacervates with KCl. At 1 mol/l, however, the layers do not separate nicely. It seems that the coacervate layer is approximately 50 % of the total volume, but the boundary stays very vague for a long time (fig. 8 *A*). A study of the refraction data reveals the cause of this phenomena. Both the coacervate and the equilibrium liquid are different from what we would expect. This means (fig. 6) that we enter the phase diagram somewhere in the middle of the curve. Here two layers will appear abruptly, each approximately of 50 %, and resembling each other very much in constitution. The slope of *E* is not parallel to *M* ("mean refraction") and "KCl" (a standard line of various KCl concentrations, parallel to M), so that we must conclude that more and more colloid is taken up into the coacervate as the concentration of KCl rises.

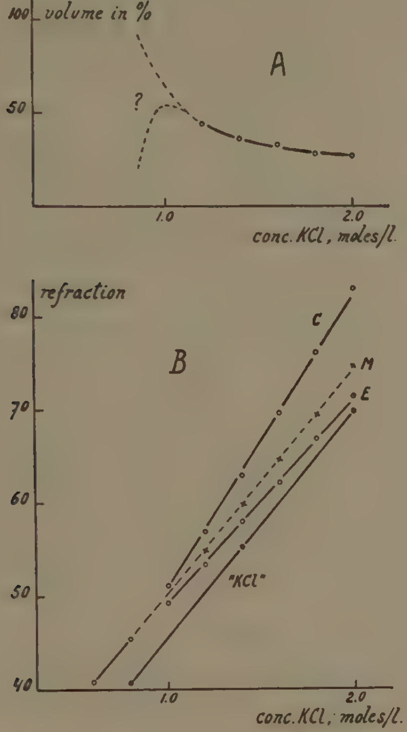

Fig. 8. *A.* Coacervate of technical sulforicinolate. It is uncertain whether coacervation starts at 100% (as in fig. 1) or at zero (as in fig. 7). — *B.* The refraction indices of the equilibrium liquids show that more and more soap is going into the coacervate layer as the KCl concentration is increased.

Summary.

1) An investigation on the refraction indices of oleate coacervate and equilibrium liquid shows that all soap is concentrated in the coacervate layers.

2) Measurements of very small coacervate layers gave very good results whereas the method of volume measurements fails in this case.

3) It is possible to calculate the volume of a coacervate layer from refraction data.

4) One can judge from refraction experiments whether colloids are present in the equilibrium liquid. This will be valuable in case of soaps which are very soluble in water (e.g. sulforicinolate). It might be expected that this method could be used with other coacervates too (e.g. gelatin — gum arabic).

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Biochemistry. — *An electrophoretical study of the proteins present in the vesicle lymph in foot-and-mouth disease and the isolation of the virus protein.* By L. W. JANSSEN. (Communicated by Prof. A. J. KLUYVER.)

(Communicated at the meeting of October 29, 1949.)

Vesicle lymph and tissue from cattle infected with foot-and-mouth disease is produced on a large scale in the European Foot-and-mouth Disease Research Institutes as raw material for vaccin production.

The lymph contains the virus of foot-and-mouth disease in relative high concentrations. When diluted one million times it is still infective for cattle. As each cow infected on the tongue yields 40 to 60 gr of this highly infective material, this represents a source of enormous amounts of virus protein.

An investigation has been made therefore of the composition of the vesical lymph originating from the tongues of cattle suffering from foot-and-mouth disease (strain *A* and *O*). At the same time an attempt was made to identify the virus of foot-and-mouth disease amongst the proteins present in the lymph.

Experimental

The infective material was taken from tongues of cattle, infected 24 h. before, immediately after slaughtering the animal and removal of the tongue. The lymph was pressed out of the minced material by means of a hydraulic press. The rough lymph was kept some time at room temperature to let it defibrinate, then it was centrifuged at 5000 r.p.m. and the supernatant filtered through filter paper until all greasy material, eventually present, had been removed.

For the electrophoresis use was made of the well-known TISELIUS electrophoresis apparatus with open electrode vessels and 4.5 cm cuvettes. Experiments were performed in a waterbath at 0° C.; a current of 50 or 30 mA was used, corresponding with a Voltage of about 200 or 140 V between both electrodes. With all experiments a phosphate buffer containing NaCl of the same composition has been applied. The protein solutions, generally in concentrations between 0.1 and 2.0 %, were equilibrated in the buffer in cellophane sausage tubes at least for one night at 5° C. The buffer contained 8.5 gr of NaCl, 0.21 gr of $NaH_2PO_4 \cdot 2H_2O$ and 2.14 gr of $Na_2HPO_4 \cdot 2H_2O$ p. l., resulting in a pH of 7.7.

The optical arrangement was that of PHILPOT–SVENSSON (1) as improved by ZERNIKE (2). Two well-corrected lenses with 9 cm diameter and a focus distance of 45 cm are present on both sides of the thermostate. The light source is a high pressure mercury valve Philips Philora HP 500, the light of which is projected on the horizontal slit by means of a simple eyeglass. The picture of the "Schlieren" is transformed by a revolving slit, an eyeglass and a cylindrical lens and is projected on the focussing-screen of a Leica revolver, which can be exchanged for a Leica camera.

The development of the pictures of both limbs of the electrophoresis apparatus is followed from the beginning by making exposures every 30 minutes, until the final picture is obtained, which mostly takes 2 to 3 hours. In this way the pictures of the ascending and the descending limb, each 24 × 36 mm, are collected on a film. A yellow filter makes the light nearly monochromatic and usually an exposure time of 15 sec gives a good picture with this arrangement.

The moderate length of the whole optical system — only 215 cm — makes for a more compact apparatus, higher light intensities and allows the measurement of lower concentrations of protein, as predicted by ZERNIKE in his paper. Even protein concentrations of 0.1 % and lower give visible diagrams.

The measurement of the relative amount of protein in a protein mixture has been made, either by counting surfaces of the enlarged diagrams on mm paper, or by measuring these with the help of a planimeter.

The absolute concentrations of proteins are measured refractometrically.

In this paper only diagrams of the descending limb are given, the ε-gradient has been neglected.

The electrophoresis apparatus as well as an ultracentrifuge of the air-driven type were built in the laboratories of the State Veterinary Research Institute after a design made by ZERNIKE and JANSSEN.

The ultracentrifuge, of the preparative, angle type, is driven by 2—5 Atm. of air pressure and runs at least 60.000 r.p.m. The rotor contains 15 celluloid tubes each containing 3 ml of fluid, it spins in a vacuum of a few mm, while the case of the centrifuge is cooled.

Results

Several electrophoresis diagrams were made of samples of lymph from animals infected with strain *A*, *O* or with a mixture of both. The concentrations of total protein in the samples of the original lymph varied from 3.5 to 6.5 %. Most determinations were made in a concentration of 1.7 %.

Fig. 1 shows the diagram of the descending limb of the electrophoresis apparatus with vesicle lymph from cattle, fig. 2 gives the diagram for cattle serum. The diagram of the lymph is representative for all lymphs investigated and no essential differences were found between lymphs of the *A* and *O* strains.

When both diagrams are compared it is immediately clear that in main outlines bovine vesicle lymph has a composition rather similar to that of cattle serum. The lymph being an exudate this is not surprising.

However, an important difference is observed in the region of the α- and β-globulins. Comparison of both diagrams shows the presence of one or two peaks which contrast with the low maxima of the α- and β-globulins of the serum.

In 1939 it was found (3) during an ultracentrifuge study of the virus of foot-and-mouth disease in Uppsala in SVEDBERGS laboratory, that vesicle lymph originating from swine contained the virus protein in large amounts. It could, therefore, be expected that the increase of proteins on the place of the α- and β-globulins is partly due to the virus protein itself, partly to other proteins, synthesized or broken down in the diseased epithelial cells.

The special features of the electrophoresis apparatus of TISELIUS allow to find out which position the virus protein may have between the serum proteins.

It was observed that in the descending limb of the apparatus filled with lymph free of haemoglobin the α- and β-globulins were slightly yellow, the same colour as shown by the virus isolated by ultracentrifugation. On the contrary the γ-globulin above these globulins was colourless.

In continued electrophoresis runs, pure γ-globulin and a mixture of serum albumin + α-globulin + β-globulin were isolated by shifting the cuvettes. The preparation of γ-globulin was found to be infectious 1 : 100 on the guinea pig; the mixture of albumin and medium globulins showed to be infectious 1 : 100.000 on the guinea pig. Thus the conclusion could be drawn that the virus did not move with a velocity of γ-globulin.

When in the same way in the top cuvettes on the ascending side serum albumin and α-globulin, on the descending side γ-globulin had been collected, the fluid present in the lower cuvettes, containing the β-globulin showed a much higher titer than that of the top cuvettes. This indicates that the virus may have about the same velocity as the β-globulin.

It has been shown earlier (4) that it is possible to precipitate the virus protein of the foot-and-mouth disease from lymph quantitatively by 40 % saturation with ammonium sulphate. Hence 30 ml of lymph were precipitated at 0° C. with 20 ml saturated ammonium sulphate. The precipitate was centrifuged, washed, dissolved in buffer, freed from sulphate by dialysing and an electrophoresis run was made.

Fig. 3 represents the electrophoresis diagram of the dissolved precipitate (34 % of the total protein). It shows no serum albumin but a large peak of γ-globulin and one on the place of the faster moving globulins. The titer of the original lymph was 1 : 300.000 on the guinea pig, that of the dissolved precipitate 1 : 1.000.000; so the precipitate contains all the virus. As the γ-globulin itself does not contain a large amount of virus it is evident that the right peak on the place of the medium globulins, in the apparatus observed to be yellow, is due to the virus.

Besides, when fig. 3 is compared with fig. 4, which diagram represents the precipitate obtained by 40 % saturation with ammonium sulphate from normal cattle serum, after washing, dissolving and dialysing, it is clear that from the lymph a larger amount of protein moving with a medium velocity is precipitated, than is precipitated from the serum. From fig. 3

a proportion of γ-globulin to medium globulins of 75 : 25 can be derived, for fig. 4 this proportion is only 89 : 11.

The gypsum adsorption method (5) applied to the protein mixture obtained by precipitation with ammonium sulphate, led to a more complete purification. The dissolved precipitate was dialysed against glycine buffer saturated with $CaSO_4$, and the virus was precipitated again at 0° C. with 20 % alcohol-ether (7 : 3). The precipitate obtained was centrifuged at 0° C. and dissolved in buffer. The electrophoretical analysis of this protein (fig. 5) showed it to be a mixture of 40 % of a yellow medium globulin and 60 % of colourless γ-globulin. The infectivity of the protein mixture on the guinea pig was 1 : 1.000.000.

Fig. 6 shows the mixture of medium- and γ-globulin obtained from another sample of lymph by 40 % saturation with ammonium sulphate at 0° C. The protein mixture is found to carry the total infectivity of the lymph. Evidently about 41 % of the mixture is medium globulin. The supernatant represented in fig. 7, after being dialysed to remove the sulphate, contains the normal serum globulins, without the high complex on the place of the medium globulins present in lymph. It was found to be practically non-infectious.

However, the best way to obtain preparations of virus not contaminated with large amounts of serum globulins is that by means of an ultracentrifuge.

Already in 1939 (3) it has been found that preparations obtained by the gypsum method and by ammonium sulphate precipitation showed a high degree of aggregation. It was impossible to use these in the quantitative ultracentrifuge, they were centrifuged down in a short time without producing boundaries.

Better results were obtained then with preparations collected by means of a Beams ultracentrifuge at about 25.000 r.p.m. (40.000 g) at room temperature during 5½ h. However, from the physical chemical point of view here too important losses of the material occur, as a great deal of the centrifuged virus protein turns insoluble.

The best soluble preparations of the virus protein of foot-and-mouth disease were made in SVEDBERGS equilibrium ultracentrifuge (6) by means of the separation cell, in which the virus was collected at 0° C. for days at 18.000 r.p.m. (20.000 g).

The aggregation of the virus found to be so strongly expressed in purified preparations cannot escape the attention of scientists working in this field. A fresh, well defibrinated, and centrifuged lymph, whether originating from cattle, or from pigs or from guinea pigs, always turns turbid when kept in the cold in a few days, losing at the same time part of its infectivity. This does not happen in a serum under the same conditions.

This tendency to aggregation makes it difficult to handle the virus, especially in the pure form. This complication was met again during the first time that the new preparative centrifuge was used in this laboratory. When lymphe was centrifuged for some hours at 30—60.000 r.p.m. (57—230.000 g) in celluloid tubes in the preparative vacuum centrifuge at room temperature, the yellow-lightbrown pellets were very difficult to redissolve, especially when the higher velocities had been used.

This made me try lower velocities of centrifugation at low temperatures (about 0° C.). It was indeed found, that under these conditions — as in the slower equilibrium ultracentrifuge — preparations of virus protein were obtained which had a reasonable solubility in aq. dest., salt and buffer solutions.

At the present time, the virus protein is isolated from the lymph by a preliminary centrifugation at 5.000 r.p.m. in an ordinary laboratory centrifuge, by a further clearing of the lymph in the ultracentrifuge 20 min. at 10.000 r.p.m. (6300 g), followed by ultracentrifugation of the supernatant for 2 h. at 20.000 r.p.m. (25.000 g) at about 0° C. After the last centrifugation the virus is present as soft pellets which, after having been washed, can easily be redissolved in salt solutions by means of a syringe having a long needle.

Several preparations of the virus proteins of foot-and-mouth disease of both strains obtained in this way were analysed in the electrophoresis apparatus. All preparations were highly infective, and showed the same infectivity as the lymph used on the guinea pig when titrated. The supernatant was usually not or only slightly infective, and could be discarded. A continued centrifugation of the supernatant at 20.000 r.p.m. did not yield a new pellet. Nor did a sample of cattle serum yield a pellet under the same conditions.

For electrophoresis protein solutions, obtained from dissolved pellets, of 0.2 to 0.7 % were used. On the average the fresh infected non-frozen epithelium + lymph of the tongue of one cow yields about 16 mg of such a preparation.

Sometimes glass disks inside the parts of the cuvettes not present in the path of light, are used to reduce the volume of the normal electrophoresis apparatus from 16 to 10 ml, in order to be as economical with the material as possible.

In the electrophoresis diagrams of these preparations only one sharp symmetrical peak is present, moving with a velocity somewhat faster than β-globulin. The peak is generally followed by a "tail" on the base line, moving with slower velocities, mainly due to electrolyte effects. As the peak is carrying the turbidity and the colour, it is necessary to remove all turbidity as far as possible by a sharp centrifugation 15 min. at 5.000 r.p.m. before starting the electrophoresis. When this is not done only one half of the peak is visible.

Fig. 8 shows the electrophoresis diagram of these preparations of the foot-and-mouth virus protein (strain *A*) after 2½ h. of electrophoresis. The *O*-strain gives an identical picture having one sharp peak with some contamination. When preparations of ultracentrifuged virus of the *A*- and *O*-strain are mixed, during electrophoresis only one single peak is observed, proving that the proteins of both strains move with the same velocity.

These preparations of the virus protein of foot-and-mouth disease are brown-yellow; they are rather unstable and when dissolved and centrifuged one day, they become turbid by aggregation or crystallization the other day. Such an instability of preparations of the virus of foot-and-mouth disease purified by filtration through graded membranes has also been reported by GALLOWAY and ELFORD (7).

In ultraviolet light, measured in the Beckmann spectrophotometer, the virus proteins of the *A*- and the *O*-strain show a strong maximum of absorption at 2590 Å due to nucleic acid. Fig. 9 represents the absorption

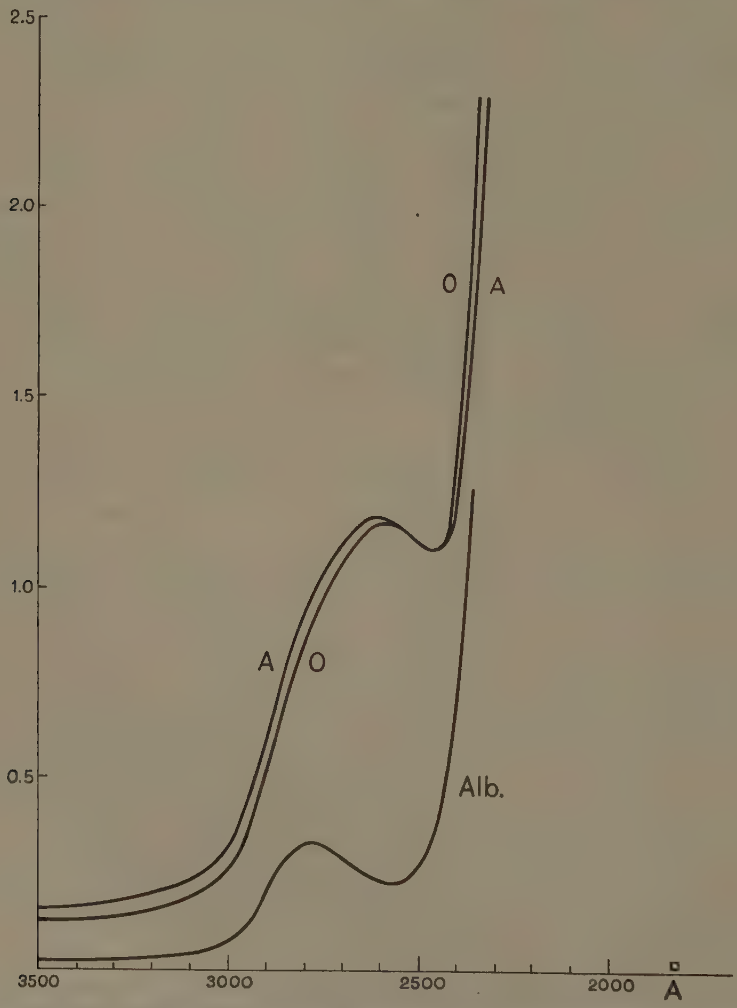

Fig. 9 gives the ultraviolet absorption spectrum of ultracentrifuged preparations of foot-and-mouth virus protein *A* and *O* and that of bovine serum albumin in a concentration of 0.033 %.

curves in the ultraviolet light of the preparations of the *A*- and the O-strain in a 0.033 % solution. Both curves only show a slight difference. At the same time the ultraviolet absorption curve is represented of bovine serum albumin (Alb.), measured in the same concentration of 0.033 % of protein. The protein, not containing nucleic acid, shows an absorption maximum at 2790 Å. The enormous difference between the diagrams of the virus proteins and the serum albumin is due to the presence of about 11.6 % of nuclei acid in the former, leading to a maximum of absorption at 2590 Å.

An estimation of phosphorus in a preparation of *A*-virus yielded 0.93 % of phosphorus, corresponding with about 10 % nucleic acid.

When bovine serum is ultracentrifuged in the same way as is done with the lymph, no pellets appear in the tubes which proves that the protein found is no constituent of the serum. However, when an extract is made of healthy epithelium of the tongue by grinding it with quartz sand, small yellow pellets are found after the same ultracentrifugal procedure. The composition of this material is the object of further research.

Discussion and conclusions

When vesicle lymph of foot-and-mouth diseased cattle is analysed in the electrophoresis apparatus, it was found that the lymph has on the whole the same composition as the bovine serum, with the exception of the presence of one important extra peak covering the α_2- and β-globulins.

The additional protein is brown-yellow and is responsible for the typical brown-yellow colour of non haemolysed lymph.

Inasmuch as fractions of pure γ-globulin and serum albumin isolated by electrophoresis have only a low infectious titer, whereas the β-globulin containing lymph is highly infectious, it is evident that the virus moves with a velocity not far from that of β-globulin.

Different procedures for precipitation of the virus protein used in the last 15 years have now been checked by electrophoresis.

When a precipitate is made in lymph by 40 % saturation with ammonium sulphate at low temperature a mixture of γ-globulin and proteins with a velocity of the medium serum proteins comes down, which mixture contains the infectivity.

Also precipitation with 20 % alcohol-ether (7 : 3) from a saturated solution of $CaSO_4$ yields a mixture of medium globulins and γ-globulin, containing the infectivity.

As the γ-globulin was shown to be only slightly or non-infectious, again the conclusion is drawn that the brown-yellow protein, moving with a velocity not far from β-globulin is responsible for the infectivity.

A third approach of the problem was made by means of the ultracentrifuge. In earlier research, it was found that vesicle lymph of pigs contained a monodisperse virus protein having a sedimentation constant $s = 18 \times 10^{-13}$. It was now found that soluble preparations of the virus protein of foot-and-mouth disease could be made in the preparative ultracentrifuge at low speed. At high speed the virus protein turns insoluble.

During electrophoresis these preparations show only one protein peak moving with a velocity somewhat faster than β-globulin.

Together with the preparations earlier obtained in Uppsala by ultracentrifugation, having only one single peak in sedimentation in the equilibrium centrifuge, these preparations of the virus protein must be among the purest preparations of this protein ever obtained.

The preparations of virus protein of the *A*- and the *O*-strain, do not show difference in electric charge. The ultraviolet absorption spectra of both are rather similar, but, compared with a normal protein, the spectrum of the virus protein indicates the presence of about 11.6 % of nucleic acid, to which can also be concluded on the amount of phosphorus present in these preparations. This result is in agreement with our findings made in 1937 (8).

Cattle serum itself does not yield a protein pellet under the conditions as described above. However, an extract of normal epithelium from cattle tongues, prepared by grinding with quartz sand does give a small yellow pellet when ultracentrifuged in the same way. Therefore, the possibility cannot be excluded that the virus protein separated is still contaminated with small amounts of a similar, non infectious protein originating from normal epithelium cells, having about the same charge and molecular weight.

As a destruction of the cells by the disease and the process of grinding with quartz sand is not at all comparable one could suggest that the virus protein of foot-and-mouth disease might originate from this normal protein by a process of biochemical breakdown. It seems, however, more probable that the virus will be synthesized on the synthesizers of the epithelium cell normally making the mentioned protein, which synthesizers, deviated by the action of the virus, start the production of the virus protein.

The work was carried out in the State Veterinary Research Institute at Amsterdam, Director Dr H. S. FRENKEL. The author wishes to thank the authorities of the U.S. Bureau of Animal Industry, Washington for providing part of the lymph used.

Summary

The composition of the vesicle lymph of foot-and-mouth diseased cattle has been submitted to an electrophoresis analysis.

In the lymph the serum proteins are found to be present in the same proportions in which they occur in cattle serum; in addition a foreign protein is present which has a brown-yellow colour and gives a peak in the diagram in the region of the α_2- and β-globulins.

From the infectivity of the different serum constituents isolated by electrophoresis it became evident that the yellow protein, moving with about the velocity of β-globulin carries the infectivity.

Purification methods for the virus protein by precipitation with am-

monium sulphate and with $CaSO_4$ and alcohol-ether, as earlier devised, yield virus preparation, which when analysed by electrophoresis, appear to consist of γ-globulin and of a protein with about β-globulin velocity.

When lymph is purified with the ultracentrifuge at low speed and at low temperature a brown-yellow protein is isolated, having the properties of the virus and moving with a velocity somewhat larger than that of β-globulin, when investigated electrophoretically. The protein is homogeneous in electrophoresis, and preparations from *A*- or from *O*-lymph move with the same velocity.

The ultraviolet absorption spectra of these preparations of virus protein of the *A*- and the *O*-strain show a maximum at 2590 Å due to the presence of about 11.6 % nucleic acid. They are about the same for both strains.

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Fig. 1. represents the electrophoresis diagram of vesicle lymph of foot-and-mouth disease. (descending limb.)

Fig. 2 gives the same diagram of bovine serum, not showing the important middle peak.

Figs 3 and 6 give diagrams of the infectious protein preparations precipitated from lymph at 0° C. with ammonium sulphate saturated for 40 %, showing a considerable amount of proteins moving with a medium velocity next to γ-globulin.

Fig. 4 gives the result of the same precipitation of bovine serum, only a small amount of β-globulin is present.

Fig. 5 shows a preparation which is first purified with ammonium sulphate and then by the gypsum adsorption method.

Fig. 7 shows the proteins remaining of the vesicle lymph, after precipitation of the infectious protein by 40 % saturation with ammonium sulphate, all serum proteins being present.

Fig. 8 shows the peak of the virus protein of foot-and-mouth disease isolated by ultracentrifugation.

L. W. JANSSEN: *An electrophoretical study of the proteins present in the vesicle lymph in foot-and-mouth disease and the isolation of the virus protein.*

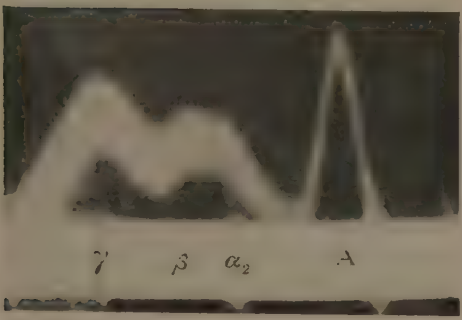
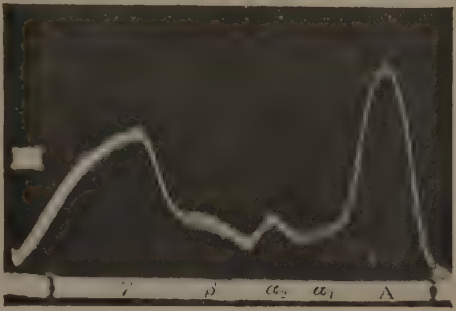


Fig. 1.



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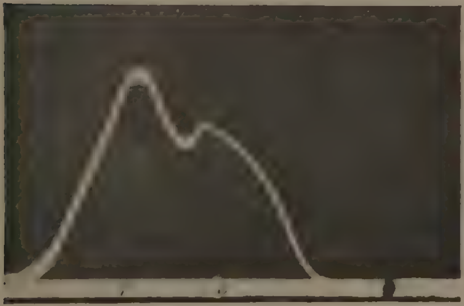
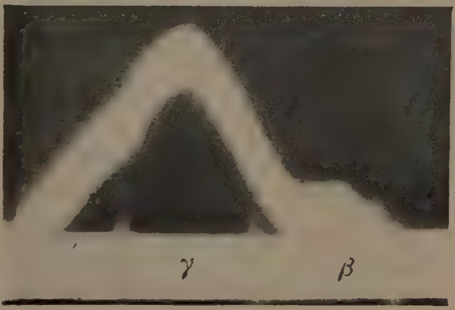


Fig. 3.



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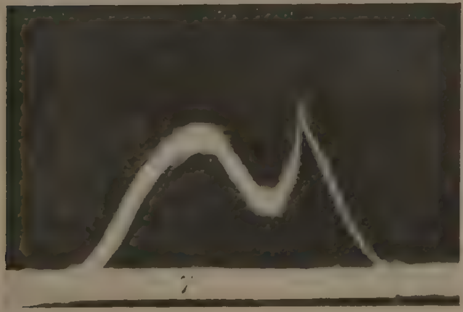
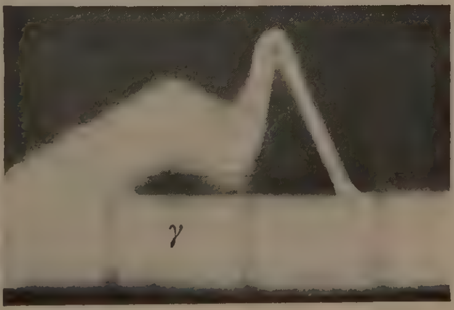


Fig. 5.



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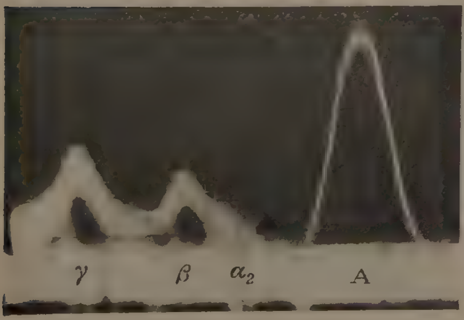

Fig. 7.

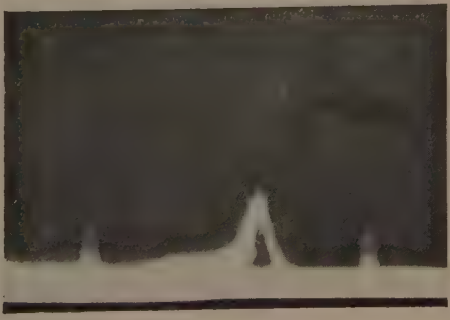

Fig. 8.

Geology. — *Diapiric structures*. By D. DE WAARD. (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of October 29, 1949.)

Abstract.

A comparison is made between the structures of salt domes and the Mt. Aigoual pluton, previously described by the author. Especially the individual structures affected by the intrusive action have been emphasized in this paper. The four most important structural phenomena show great resemblance in both diapiric forms. They point to a similar mechanism of intrusion of the atectonic salt domes and the posttectonic Mt. Aigoual pluton.

Introduction.

The word "diapir" (from the Greek: *διαπείρω*, to pierce) has been introduced by MRAZEC for anticlinal salt intrusions in Roumania. The term is used afterwards in broader sense, including equivalent occurrences in other sedimentary and igneous rocks. The mechanism of the occurrence or "diapirism" caused a number of structural phenomena which are characteristic for the diapirs of different composition.

Since long the diapirism of igneous rock and salt have been the subject of comparative study. At first the behaviour of salt was elucidated by comparison with the intrusive action of granite. Since the rapidly increasing knowledge of salt domes by economic research and laboratory experiments, it proved however to be fruitful to consider the behaviour of igneous rocks as reflections of similar effects in salt.

Usually comparisons are more or less theoretical. WEGMANN (1930) especially pointed out the structural origin of granite intrusions in orogens which he regarded to be comparable with the development of salt domes. Generally little attention is paid to the resemblance of the individual structural phenomena caused by diapirism. They will be especially emphasized in this paper.

During the survey of the Mt. Aigoual pluton the author had the opportunity to examine in detail many clearly developed features, which designates this pluton as a characteristic example of an igneous diapir (DE WAARD, 1949 *a, b*). In the following paragraph its tectonic outline will be briefly described.

The Mt. Aigoual pluton.

The Mt. Aigoual pluton is situated in the Cevennes, the southeastern part of the Central Massif in France. The pluton is a small stock-like posttectonic granite intrusion in probably Cambrian slates, which were folded in the Sudetic phase of the variscan orogeny.

The massif has a, generally very homogeneous, granodioritic composition, with large phenocrysts of orthoclase. In a corona round about the massif, particularly north and east of it, numerous granite porphyry dikes have been mapped. Besides those, quartz prophyry, tonalite porphyry, pegmatite, aplite and lamprophyre dikes have been found as well in the pluton outcrop as in the country rock. The contacts of the pluton are very sharp and a narrow faintly developed contact zone is usually shown.

The intrusive character of the granite is very well expressed by flow structures. Flow planes as well as flow lines of oriented phenocrysts are generally in conformable arrangement. More in detail a complicated structure is shown, in which at least three individual domes may be distinguished (fig. 1).

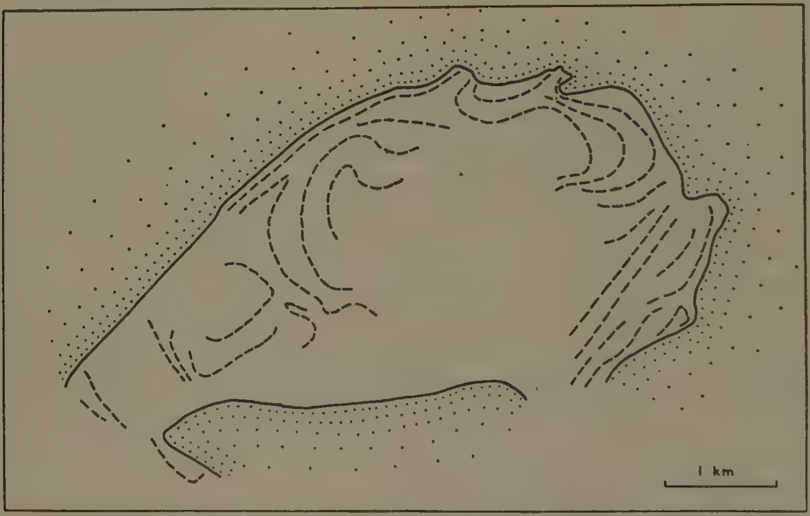

Fig. 1. Planar flow structures in the Mt. Aigoual granite.

The directions of the contact planes of the pluton, the dikes and the faults in the country rock proved to coincide with the joint systems of the country rock, which are found to be due to preceding orogenic movements. The intrusion must have forced its way upward by using mostly these existing joints. The country rock has been transformed in a block faulted area; blocks of slates seperated by granite porphyry dikes north of the pluton outcrop indicate the presence of the pluton body underneath (fig. 2).

Interesting evidences of the diapiric character of this pluton are the nearby and long-distance effects of the intrusive motion. In a zone up to about 500 m. round about the pluton the country rock proved to be bent upwards, which is shown by a considerable increase of the normal dip of the slates and curving of the normal strike to the N.E. (fig. 3).

Contortion of the strike combined with increase of the dip may be simply the result of the bending upward of inclined layers as is explained in the theoretical diagram (fig. 4). This bending upward of the country rock near the contacts can only be interpreted by the frictional effect of the intruding mass upon its walls.

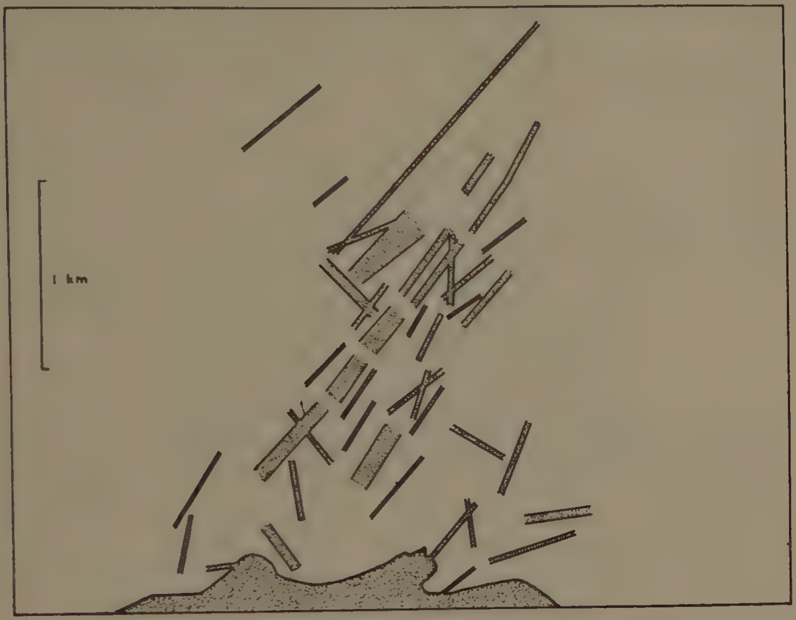

Fig. 2. Dike swarm north of the outcrop of the pluton.

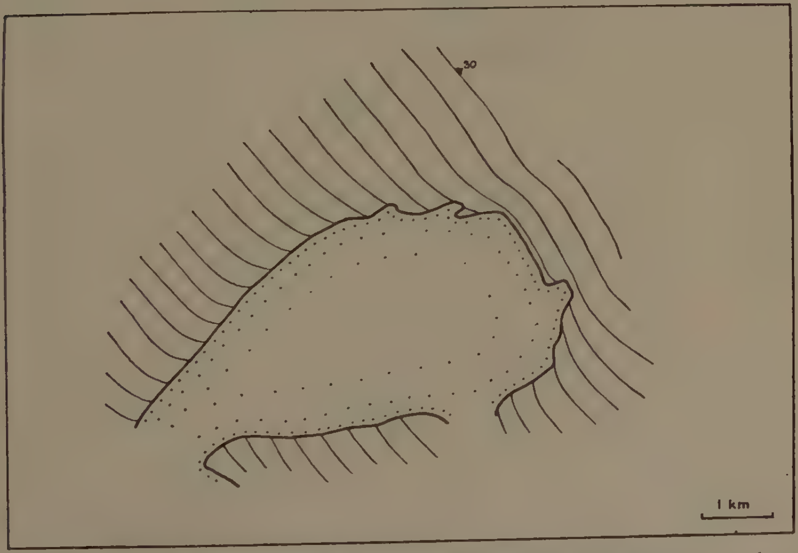

Fig. 3. Contortion of the schistosity planes in the country rock adjacent to the pluton.

Long-distance effects are evident as well. Up to 6 km from the pluton in the mapped area, faults more or less parallel to the nearest contact have been observed, most of them with drag features indicating uplift of the country rock near the pluton and a relative downwarping at the other side (fig. 5). As a whole the country rock round about the pluton seems to have moved upward in blocks, which can only be explained — being

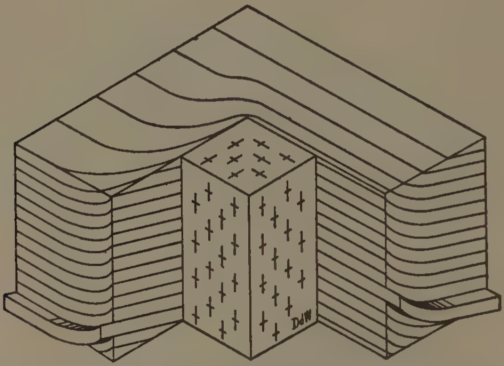

Fig. 4. Block diagram illustrating the strike and dip alterations by bending of the schistosity planes, after DE WAARD (1949*b*).

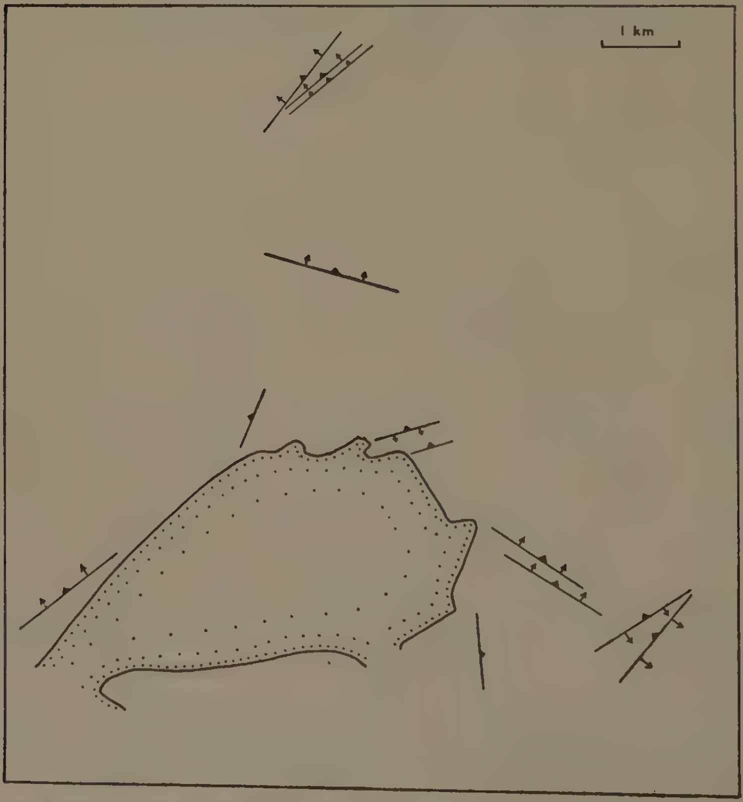

Fig. 5. Dragged faults in the country rock. Arrows indicate the downthrown sides.

a posttectonic intrusion — as a result of the upward motion of the granitic mass.

The structural investigations in and around this massif proved firstly the existence of a magma or mush by its clear flow structures and its effects on the surrounding country rock, secondly the intrusive character of this mass by its dome-shaped flow structures and thirdly the power of its upward motion by the contortion of the bordering country rock and the uplift of country rock round about the pluton in this area. Emphasizing these conclusions is necessary in this time of controversy in petrogenesis.

The mentioned conclusions only refer to the Mt. Aigoual pluton, but identical structures observed in neighbouring massifs of the Cevennes indicate a similar origin.

Structural comparison of granite and salt diapirs.

As a rule salt cores, originated in ideal circumstances, approximate a cylindrical shape. They widen downwards into their basic layer and show sometimes a tendency to be funnelshaped on top. Their shape largely depends however on the structure of the invaded rocks.

Especially posttectonic and atectonic granite massifs may have rounded outcrops, but more than in salt domes, complicated pre-existing structures of the intruded rock are responsible for their outline. The Mt. Aigoual pluton for instance has an elliptical outcrop, which is largely effected by existing joint systems of the slates.

According to the structural data of salt domes, the most conspicuous diapiric phenomena may be summarized as follows:

1° folded and doming salt layers within the core,

2° dragging upward of the sedimentary rocks adjacent to the core,

3° synthetic faulting in the country rock,

4° tension joints in the sediments overlying the core.

Identical structural phenomena have been found to be the most important of the Mt. Aigoual pluton.

To a certain extent folded salt layers are comparable with flow structures in granite. Differences appear when considering flow pattern reconstructed by oriented crystals such as in the Mt. Aigoual granite. The orientation of the phenocrysts is the result of the deformation of the mass; folded salt layers however are the product of sedimentation combined with the deformation. The former shows the actual direction of flow, whereas the latter gives a picture of the deformation of a horizontal layered rock. Both structures approximate however the direction of motion during the process of deformation.

Though a close comparison is not possible, general similarities are evident. The German salt core of fig. 6 shows a conformable pattern by strongly folded salt layers of different composition. They are dome-shaped and often vertically folded, curved inward as well as towards the walls. In detail the kulissen folds may be very complicated (fig. 7).

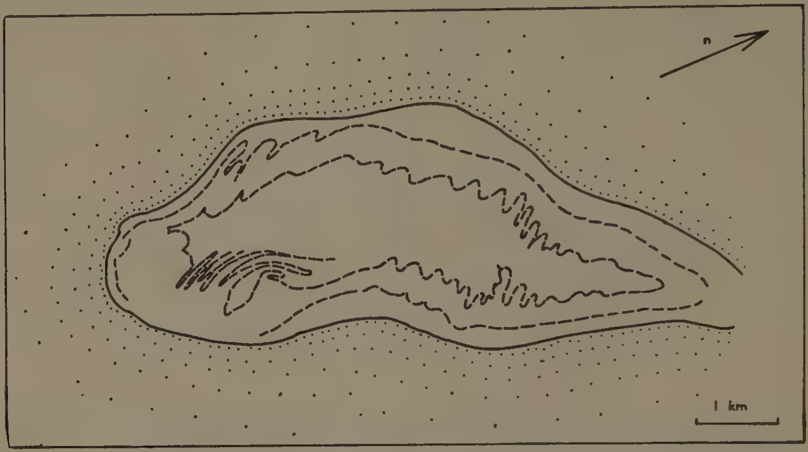

Fig. 6. Flow structures shown by deformed salt layers in the Benther salt dome, Germany. Reconstruction in a horizontal section at about 500 m, after STIER (1915).

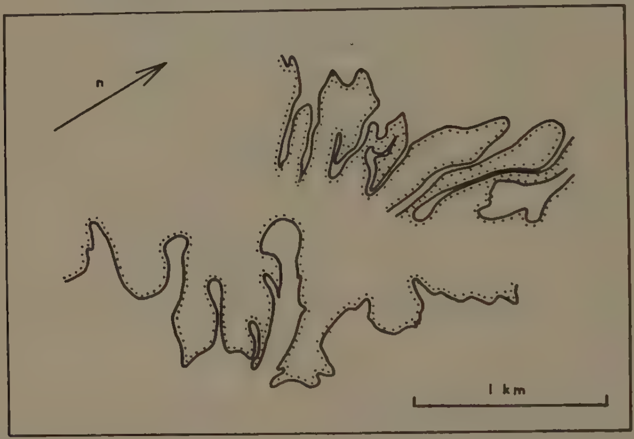

Fig. 7. Complicated folding in a single salt layer. Detail of the Benther salt dome of fig. 6, in a horizontal section at 601 m, after LOTZE (1938).

In the Mt. Aigoual pluton most platy flow structures are also conformably arranged. Only in some places flow planes seem to abut at different angles against the contact planes. This will probably be the result of the above mentioned difference by measuring the orientations of crystals instead of pre-existing layers. The conformable structures and the folded or dome-shaped pattern inside is very similar to salt core structures. As they represent the directions of the actual flow, the continuity in the reconstructed flow lines fails. This continuity — except for adventitious actions, e.g. rents — is characteristic for the pattern in salt domes.

The many borings in the salt dome oil fields give a clear picture of the dragging upward of sedimentary rocks adjacent to the core. Especially in the Gulf Coast area, where salt masses pierced through and warped flatlying sediments, this phenomenon is the strongest evidence of diapirism. In most salt dome regions, preceding or synchronus orogenic movements have complicated the diapiric structures.

The surface map of one of the American salt domes (fig. 8) illustrates the extent of the area affected by uplift. The flatlying sediments are inclined within a zone of about 800 m width from 1° up to 20° near the outer margin of the core. The following list — according to the sections given by POWERS (1926) — shows the width of the zone affected by pure drag in some Gulf Coast salt domes in connection with the diameter of the core:

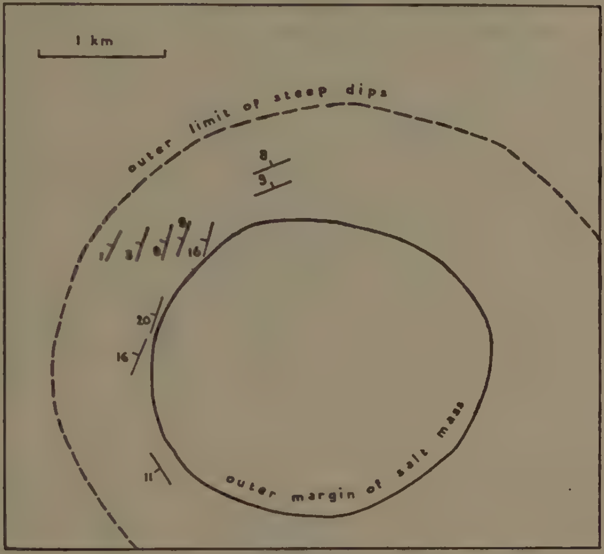

Fig. 8. Surface map of the Grand Saline salt dome, Texas, illustrating the area affected by drag, after POWERS (1926).

Salt dome:	Diameter of the core in km:	Width of the zone in m:
Grand Saline salt dome	2.4	800
Steen salt dome	1.9	400
Brooks salt dome	2.4	900
Keechi salt dome	1.3	650
Palestine salt dome	2	950
Butler salt dome	1.6	650

The drag phenomenon is as clearly developed around the Mt. Aigoual pluton (fig. 3). The affected zone is here recorded by alterations of strike and dip of the slaty cleavage, which is nearly constant in this area. Alterations are observed in a zone of 500—700 m width, nearly everywhere round about the pluton outcrop. Notwithstanding the important differences in the circumstances of salt and granite diapirism and in the properties of invading masses and invaded rocks, the width of the affected zone is here of similar size as in the mentioned salt domes.

As the dislocation in the normal strike and dip of the slaty cleavage caused an alteration of the strike as well of the dip (fig. 4), the exact bending of the normal cleavage planes can not be determined directly. For that purpose all strikes and dips measured within this zone have been rotated in pole diagram (fig. 9) in the same way as the average of the normal cleavage (A) towards the centre (A_1). The diagram thus illustrates the divergences with respect to a horizontal cleavage plane. Up to 50° bending has thus been caused by the frictional effect of the upward moving mass upon its walls.

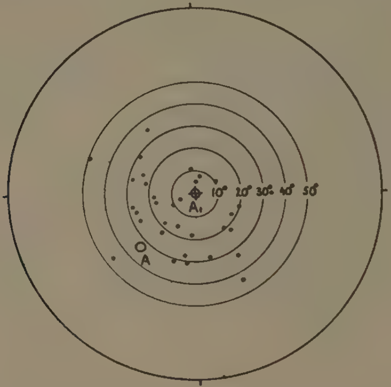

Fig. 9. Strike and dip divergences caused by drag in the country rock adjacent to the pluton. Pole diagram of the slaty cleavage after rotation of the average of the normal schistosity of the slates (A) towards the centre (A_1). The poles represent the direction and the actual amount of bending of a horizontal schistosity plane.

Sections through salt domes usually show a bending of the layers approaching an arc of a circle, the amount of uplift of the strata approximates the width of the affected zone. Applying this approximation to the Mt. Aigoual pluton, this would mean an uplift by bending of the country rock near the contact of about 500 m.

Closely related to the dragging of layers is synthetic faulting in the neighbourhood of the diapir. The upward mechanism, which is responsible for the bending of the layers locally caused rupture and dislocation. The country rock gives way, partly by bending and partly by faulting, dependent on the properties of the rock. The two structural phenomena co-operate with each other, both resulting in an uplift of the country rock adjacent to the diapir.

Dependent on the nature of the rock and on the shape of the diapir, faults may originate close to the walls in the dragged area or at greater distance (fig. 11). In both cases layers are gradually faulted up towards the diapir. Nearby synthetic faults are illustrated in several sections through

American salt domes. Their strike and dip usually are parallel to the nearest contact. In the ideal case they would exhibit a concentric arrangement around the core, cutting the country rock in large tubes.

Round about the Mt. Aigoual pluton, both nearby and long-distanced faults are very well developed (fig. 5). In most of them drag features indicate an uplift of the country rock at the side of the granite outcrop. Though these faults are largely originated from pre-existing joint systems in the slates, they may be called synthetic as well; they are caused by and have co-operated with the diapiric movements in this area.

As a result of the upward movement, the sediments overlaying the diapir are arched and often broken by tension. Ideal sections show the tension joints parallel to the radii of the bending (fig. 11). The horizontal pattern of tension joints may be in ideal cases radialy, but here again the structure pattern is highly dependent on the pre-existing structure of the invaded rocks and on the shape of the diapir.

The structure on top of salt domes may show a random, a radial or a parallel arrangement of joints, which are usually accompanied by faulting. Fig. 10 is an example of an American salt dome with a parallel fault system.

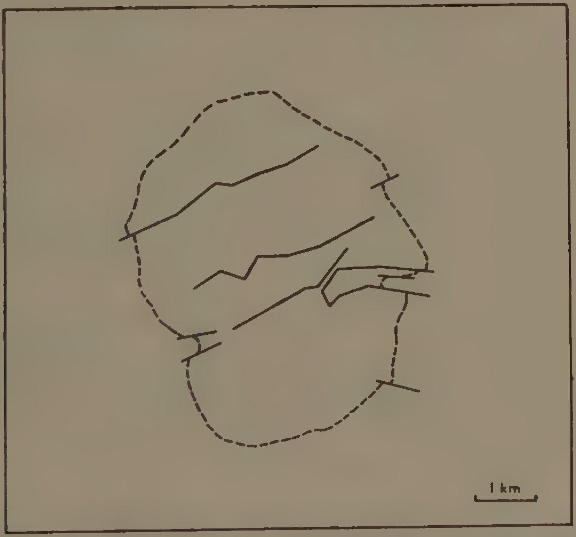

Fig. 10. A system of parallel faults caused by tension, at about 1500 m depth in the arched roof of the Conroe salt dome, Texas, after MICHAUX and BUCK (1936).

Tension joints in igneous diapirs are known in granite tectonics as "cross" joints. They have been found in the upper parts of the diapir at right angles to the flow structure, which is the direction of maximal lengthening in the mass, usually being parallel with the contact of the roof (fig. 11). Cross joints cross the contacts of the massif and penetrate the overlaying country rock. They are due to tension in het country rock and in the consolidated outer shell of the — still active — diapir.

Being perpendicular to the usually dome-shaped flow structures, the associated cross joints in sections often show a fan-shaped system. They may be filled with dikes and form dike swarms.

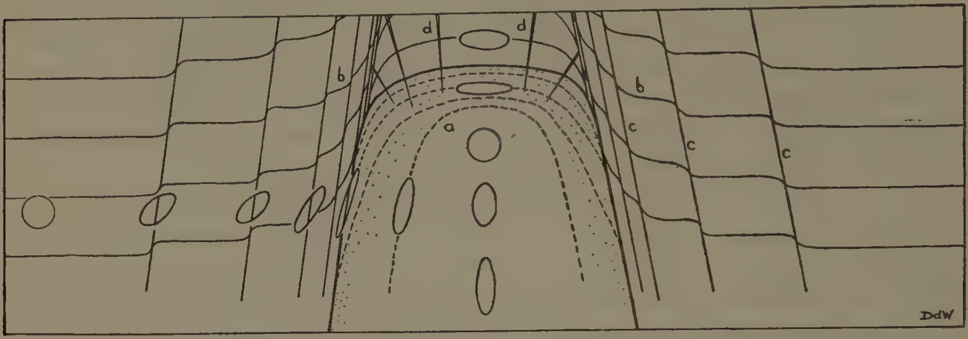

Fig. 11. Idealized section through a diapir and its country rock, illustrating the position of the strain ellipsoid in relation with the principal structures: *a*. flow parallel to the longest axis, *b*. dragged layers inclined to the longest axis, *c*. synthetic faults along one of the planes of maximal shear, and *d*. tension joints perpendicular to the longest axis of the ellipsoid.

North of the Mt. Aigoual pluton an area with numerous dikes of granite porphyry has been mapped (fig. 2). They are mostly parallel in a direction which proved to coincide with one of the pre-existing joint systems in the slates. These dikes indicate a considerable distension of the rock perpendicular to their general direction. This must have been originated by arching of the area during the upward movements of the granite mass underneath, associated with tension in the roof and opening of pre-existing joints at right angles to the strongest bending, followed by filling up with granitic material.

The combined thickness of all dikes indicate a distension of this area of about 400 m per 900 m of the recent surface. This would mean an uplift of about 370 m in the central part of a corresponding arch. An uplift of about 500 m maximal has been roughly estimated from the bending of the dragged wall rock.

In this paragraph some of the most important structures have been discussed. They are common phenomena in salt domes as well as in plutons. Though the properties and the behaviour of the rocks in salt domes and plutonic areas are very different, and in spite of complications by pre-existing structures, the mentioned phenomena proved to be comparable in many ways. This indicates a great resemblance in the mechanism of emplacement of these diapirs.

The posttectonic Mt. Aigoual pluton as well as the mentioned salt domes have been active, not or to a very small extent related to synchronous tectonic movements. In the same way there may be expected a similarity in the structures of synorogenic granites and tectonically affected salt domes, both being originated by diapiric activity associated with orogenic movements.

Conclusions.

Diapirism is a more or less independant mechanism in the earth's crust. The piercing mass, being the bearer of the intrusive force, effected differential movements at both sides of the walls and tension in the roof. These accompanying effects caused the four structural phenomena, mentioned in the preceding paragraph. They are the principal structures, which characterize the diapir.

Though affected by pre-existing structures, the four principal phenomena are clearly developed in the Mt. Aigoual pluton, emphasizing its diapiric origin. It does not seem probable, that the Mt. Aigoual pluton would be an exceptional case among granite plutons. Similar structures have been found in the neighbouring massifs, indicating the same origin.

The granite tectonic investigations of the last decades proved that diapirism of granite must be regarded a common feature and an important part of the granitic behaviour. Upward moving batholiths have become familiar by the publications of HANS CLOOS. He summarized his vision of the emplacement of batholiths in illustrations of large streamlined and round-topped bodies, penetrating the upper crust with root-like connections in deeper layers (see e.g. CLOOS, 1936).

WEGMANN (1930) proposed a possible explanation of the structural origin of granite diapirs. Like a salt dome a layer of highly mobile material underneath a more rigid zone of the earth's crust, presumably of lower specific gravity, stores up in one place and moves upward. As is supposed for salt domes an initial impuls is necessary. WEGMANN mentioned the basement folding of ARGAND which would give rise to the origin of granite diapirs in orogens. Once started, the directional pressure gradually exceeds the pressure from all sides and upward motion gradually increases in upper zones.

This interesting comparison leads to the necessity of a source layer as salt domes have. From this unknown source the diapir originated and started under favourable circumstances to penetrate into higher levels of the earth's crust. The source layer, zone, or chamber may have been formed by magmatism, migmatism, metasomatism, etc., according to the conception one supports.

To expose the source zone, the country must be deeply eroded and this will only be the case in relatively few regions. Hence, most areas show only the diapiric extracts of this zone. A rough approximation of the percentages of granite types has recently been mentioned by GROUT, (1948):

> "I think more than 85 per cent of granites exposed were formed by cooling magmas (magmas not necessarily 100 per cent liquid)."

The other 15 per cent or less, originated by metasomatism, emanations and migmatism, may thus represent the exposures of the source zone.

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Botany. — *On the influence of pH on the growth of Avena coleoptile sections.* (Preliminary note.) By J. RIETSEMA. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of September 24, 1949.)

Introduction.

STRUGGER (1932, 1933) was the first to investigate the influence of pH on plant growth in connection with the action of growth substances. He found that the growth of roots and hypocotyls of *Helianthus* in Na-acetate buffers showed two pH-optima, namely at pH 4.25 and 5.5, whereas at pH 5.1 growth was minimal. The relation of this phenomenon to growth substances was supposed to be an indirect one.

J. BONNER (1934) failed to reproduce two pH optima for "acid-curvatures" of *Avena* coleoptiles. According to this author, the curve obtained by plotting the "acid-curvatures" against the internal pH of the coleoptiles closely equals the dissociation curve of the growth promoting substance produced by *Rhizopus suinus* (DOLK and THIMANN (1932)). The internal pH of the cells was supposed to be changed by the external solution.

A. M. A. VAN SANTEN (1938, 1940) confirmed this fact in studying the straight growth of small sections of *Avena* coleoptiles and of roots of *Pisum sativum* and *Helianthus annuus*; the "growth-external pH" curves showed a close similarity to the dissociation curve of both auxin-a and indole-3-acetic acid (I.A.A.). Moreover Miss VAN SANTEN (1940) stated, that STRUGGERS results are disputable, since the condition of the plants used was unsatisfactory and Na-acetate buffers are toxic to roots of *Helianthus*. SAKAMURA and KANAMORI (1935) showed the toxicity of Na-acetate buffers to root hairs of *Brassica*.

The pH of the cell contents was assumed to be changed by the external solution and to obtain the value of the latter gradually, whereas growth is exactly proportional to the concentration of undissociated molecules of the growth hormone present inside the cell.

D. M. BONNER (1938) revealed the fact that several synthetic growth substances are equally active in the pea test, with regard to the concentration of undissociated molecules only. Also it was stated, that the internal pH of the tissue was changed by the external buffer solution. This internal pH was measured with a glass electrode after crushing the tissue. By plotting these pH values against the curvatures a graph was obtained showing a close correlation with the dissociation curves of the growth substance used. On the other hand KÖGL, HAAGEN SMIT and ERXLEBEN (1933) discovered that the activity of solutions of auxin-a in the *Avena* test is improved by the addition of small amounts of KCl and acetic acid. It was shown by DOLK and THIMANN (1932), that indole-acetic acid, produced by

Rhizopus suinus enters the cell in the undissociated state only. ALBAUM, KAISER and NESTLER (1937) proved this to be true for the diffusion of indole-3-acetic acid molecules into the vacuole of *Nitella* cells. It may be dangerous, however, to apply these results to epidermal cells of a coleoptile, since *Nitella* consists of threads of single cells, whilst the coleoptile is a real tissue.

In 1938 THIMANN and SCHNEIDER succeeded in demonstrating the same phenomenon in experiments concerning the accelerating action of I.A.A. on protoplasmic streaming in epidermal cells of *Avena* coleoptiles.

All these experiments were of short duration. In prolonged experiments (24 hours) Miss VAN SANTEN (1938) obtained the same results. However, no full report of the results was given and the figures presented are unsufficient to substantiate any conclusion.

Summarizing this brief survey of literature we may state, that two different actions of a buffer solution have been taken into account:

I. Action within the cell: Growth is determined by the external pH, which changes the pH of the cell contents. Only the undissociated molecules of the total amount of growth substance present inside the cell are promoting growth.

II. Action outside the cell: By adding growth substance to the external solution growth is promoted only by those molecules diffusing into the cells. The number of these molecules is determined by the external pH, for only undissociated molecules are able to penetrate into the cells.

According to VAN SANTEN (1940) both actions may cooperate, but in *Avena* coleoptiles not in the sense of a mere addition.

HITCHCOCK and ZIMMERMAN (1938) raised serious objections against these views. They claim to have shown that the effect of buffer solutions on the root formation in leaf cuttings of tomato plants merely depends on the molarity of the buffer solution used and not on its pH. High activity was obtained always at complete dissociation of the growth substance. And, since the assumption that the internal pH can be lowered by external buffer solutions, never has been proved, it is concluded that the dissociation hypothesis has neither application to root formation in tomato leaf-cuttings, nor to other material.

POHL (1948), refuting the dissociation hypothesis, supposed both an increase of the hydrogen ion concentration and addition of indole-3-acetic acid to raise the growth of *Avena* coleoptile sections by increasing the water permeability.

It is evident that the problem of the pH effect on growth needs further investigation. First we present some experiments dealing with the effect of external buffer solutions on the pH of the cell contents; second some results concerning the influence of the external pH on the growth promoting activity of some growth substances are given.

II. *Material and methods.*

The "Victory oat" strain from Svalöf was used as a test organism. Three days old coleoptiles grown in complete darkness in an air-conditioned room at 23° C and 96 % relative humidity were cut into small sections of $1\frac{1}{2}$ mm, 5—$6\frac{1}{2}$ mm from the tip, with a microtome after VAN DER WEY (1932). For each pH value at least one series consisting of 20 sections was used per experiment.

According to VAN SANTEN (1938) the sections were mounted on a slide by means of a narrow strip of pure vaseline. The slides were placed into a petri dish with 100 ml buffer solution. The length of the sections was measured under a microscope at low power magnification in orange light. Before and after measuring the sections remained in complete darkness. During the experiment the solution was aerated intensely. In this way well reproducible results were obtained.

The buffer solutions were prepared of mixtures of 0.01 m KH_2PO_4 and 0.01 m K_2HPO_4. As VAN SANTEN showed a toxicity of Na-ions for roots only potassium salts were used to avoid any deleterious action of Na. The pH measurements were performed with a quinhydrone electrode.

The synthetic growth substances used were an indole-3-acetic acid preparation from KAHLBAUM, Berlin, and one from the Amsterdamsche Chinine Fabriek. Both indole-3-acetic acid samples had the same acivity pro mg.

pH measurements of the cell contents were performed by means of a micromanipulator after DE FONBRUNE (1937). After immersion during 24 hours in a buffer solution a section of a coleoptile was washed twice with distilled water and was mounted on a slide in such a way, that it could not be pushed aside by the micropipet while piercing the cell wall. Immediately after adjusting the section a small drop of indicator solution (prepared after J. SMALL (1928)) was injected into the cell and the colour was observed. The two main colours of the indicator on the acid and on the alkaline side were taken into account only. Intermediate colours were not taken into consideration (SMALL's "Range Indicator Method"). By using various indicators it became possible to estimate the pH between two limits. Table I (taken from J. SMALL (1928), chapter 8) shows the indicators used and the pH limits for the principal colours.

III. *Experimental.*

A. pH of the cell contents.

In order to determine the pH of the cell contents several methods have been used by various investigators. Only a few of them applied the direct method of injecting indicator solutions into the cells.

J. BONNER (1934) and D. M. BONNER (1938) crushed the tissue and measured the pH of this material after adding som distilled water.

The following considerations may show that this method is not reliable:

a. Crushed tissue consists of a mixture of cell walls, disorganized pro-

TABLE I. Characteristics of the indicators used.

Indicator	Alkaline Colour Range	pH	Acid Colour Range	pH
Bromo-cresol purple	pale blue to deep purple	6.2	yellow	5.9
Methyl red	yellow	5.6	pale pink to deep red	5.2
Benzene-azo-a-naphthylamine	yellow	4.8	pale pink to deep red	4.4
Bromo-cresol green	pale green to deep blue	4.4	yellow	4.0

toplasm and vacuole contents. Since these constituents in a living cell each may have a different pH, the significance of the pH determinations of crushed tissue is doubtful. *b.* After crushing most of the CO_2 formed by the cell metabolism will disappear since in a living cell the CO_2 pressure markedly exceeds that of the open air. As CO_2 lowers the pH, crushing causes a pH increase of the cell contents (SMALL, 1928). *c.* Disorganizing protoplasm often results in a sudden decrease of the pH, as has been observed by CHAMBERS (1932).

The *Nitella* cells, ALBAUM et al. (1937) worked with, are more favorable material. The vacuole contents of a number of cells were collected and the pH was determined by means of a glass electrode. As to this procedure only the objection under *b* remains. No pH changes of the vacuole contents were observed, even if the cells remained in a buffer solution during one hour. But as to the pH of the protoplasm no information was obtained.

The above mentioned objections are avoided by determining the pH by means of vital staining, but unhappily most indicators are no vital dyes. Consequently they do not penetrate into the living cell (GUILLIERMOND, 1941).

A more exact method consists of microinjection of indicator solutions, as described by REISS (1926), SMALL (1928), PLOWE (1931), CHAMBERS and CAMERON (1932).

The present experiments showed that the pH of the cell contents remains constant throughout a period of 24 hours in a 0.01 m phosphate buffer solution. Though it was difficult to prevent a disorganization of the protoplasm, followed by typical changes shown in the cell structure after killing caused by too much stirring of the micropipet, we succeeded in a number of cases in observing the indicator within the cytoplasm. So it proved to be possible to estimate the pH of the vacuole and the cytoplasm separately. Table II summarizes the results of several experiments.

It must be emphasized however that these results are preliminary; the experiments will be continued and extended.

As can be clearly seen from table II the pH of a living cell remains constant during a period of 24 hours. This holds even for tissue in a toxic

TABLE II. Influence of the pH of the medium on the pH within the cell.

External pH	Indicator	Cytoplasm		Vacuole	
		Colour	pH	Colour	pH
4.2	Bromo-cresol purple	yellow	<5.9	yellow	<5.9
	Methyl red	yellow	>5.6	yellow	>5.6
5.8	Bromo-cresol purple	yellow	<5.9	yellow	<5.9
	Methyl red	yellow	>5.6	yellow	>5.6
8.3	Bromo-cresol purple	yellow	<5.9	indiff.*)	5.9-6.2
	Methyl red	yellow	>5.6	yellow	>5.6
3.4**)	Methyl red	indiff *)	5.2-5.6	red	<5.2
	Benzene-azo-a-napht. am.	yellow	>4.8	indiff.*)	4.4-4.8
	Bromo-cresol green	blue	>4.4	blue-green	>4.4

*) Indifferent means, that the colour was vague and intermediate between the two principal colours.

**) The coleoptile sections had lost their turgescence and may be considered dead.

buffer solution of pH 3.4 in which both cytoplasm and vacuole showed a considerably higher pH than the surrounding solution. In a living cell both cytoplasm and vacuole show pH values between 5.6 and 5.9.

These results are to be expected from the reviews on this subject by SMALL (1928, Chapters 16 and 17).

The constancy of the pH of the vacuole can be understood from considerations by ARISZ (1945). This author concluded from theoretical arguments as well as from dates derived from both literature and own experiments that, whilst the protoplasm of a cell is permeable to inorganic ions and aminoacids, the permeability of the tonoplast is different.

Now, the pH of the vacuole seems to be of minor importance for growth in contrast to the pH of the protoplasm. But even the pH of the protoplasm proved to remain constant. We must take into consideration however that the pH measurements of the protoplasm only bear upon the inner layers. The method used gives no information on the outer layers.

The mere fact however, that the hydrogen-ion concentration influences growth, means that in the protoplasm changes in the structure must have taken place. These changes most probably have been caused by a new ion-equilibrium between the buffer solution and the protoplasm (ARISZ, 1945).

Not yet published experiments on protoplasmic streaming give us further information on the influence of buffer solutions on the inner protoplasmic layers. They have shown that the rate of protoplasmic streaming in epidermal cells of a coleoptile remains fairly constant after a 24 hours stay in a 0.01 m phosphate buffer of various pH. It seems highly improbable that changes in the ionic composition of the protoplasm are not accompanied by changes in the rate of protoplasmic streaming. Vid. STRUGGER (1926, 1928), COLLA (1929), SEIFRITZ (1943, p. 78).

If the dissociation theory with regard to the natural growth substance present inside the cell is true, this implies that the reaction of the medium as concerned with growth is limited to the outer layers.

In this respect WENT's assumptions (1938) that the action of indole-

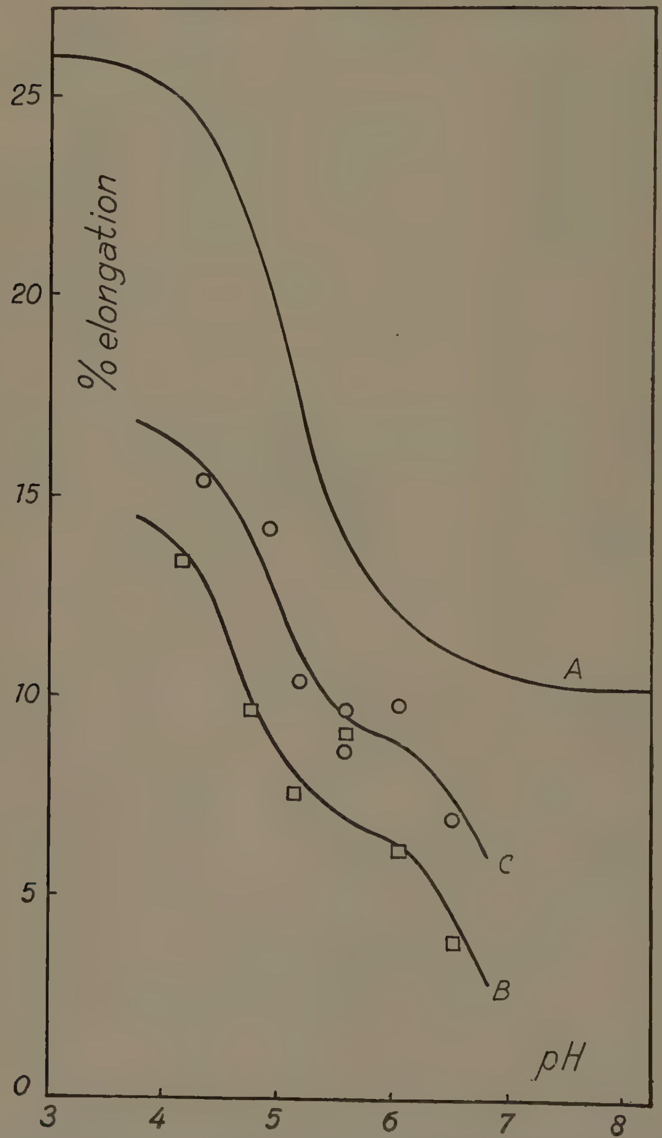

Fig. 1. Relation between the growth of *Avena* coleoptile sections and the pH of the medium. Curve *A* after VAN SANTEN (1938). Curves *B* and *C* original.

3-acetic acid on protoplasmic streaming takes place at the surface of the protoplasm may be of importance. The effect of the natural auxin on actual growth may proceed there as well.

B. Experiments on growth.

§ 1. *The influence of the hydrogen ion concentration.*

In accordance with Miss VAN SANTEN's results (1938, 1940) growth after 24 hours proved to be optimal at pH 4.2 and to decrease towards higher pH values. Compared with her experiments the growth rate is lower and more variable. This is illustrated by fig. 1. Curve *A* represents the relation between growth and pH after VAN SANTEN, whereas curves *B* and *C* are representing two out of our concordant experiments. Moreover, from pH 6.5 growth decreases more rapidly. The cause of the greater variability still remains obscure.

§ 2. *The influence of pH and glucose.*

Addition of glucose to the medium increases growth to a considerable extent, independent of the pH of the surrounding solution, as is shown in fig. 2 and table III.

TABLE III. Influence of pH and glucose on the growth of *Avena* coleoptile sections.

pH	% Elongation after 24 hours		Difference
	control	½ % glucose	
4.1	13.3 ± 0.8	21.7 ± 1.0	8.4 ± 1.2
4.8	9.7 ± 0.4	20.6 ± 0.6	10.9 ± 0.7
5.1	7.6 ± 0.5	16.4 ± 0.7	8.8 ± 0.7
5.6	9.1 ± 0.4	20.1 ± 0.9	11.0 ± 1.0
6.1	6.2 ± 0.5	15.0 ± 0.7	8.8 ± 0.9
6.5	3.9 ± 0.4	12.6 ± 0.9	8.7 ± 1.0

Consequently two processes are involved in growth: a "glucose process" independent of pH and, moreover, a pH sensitive process. The effect of both processes accumulate.

Experiments on the relation between growth and the glucose concentration have made clear that ½ % glucose is suboptimal. Hence in our experiments the glucose concentration is the factor limiting growth. Since the medium does not change the effect of glucose, we may conclude for the present that neither the permeation of glucose into the cells, nor the process in which it is involved, is influenced by the surrounding solution.

§ 3. *The influence of pH and growth substance.*

It was stated in the introduction that the experiments by VAN SANTEN (1938) on the influence of the hydrogen ion concentration on the action of indole-3-acetic acid need extension to conclude to an effect of the pH with certainty. She showed that the growth rate was affected in buffer solutions to which indole-3-acetic acid had been added in a concentration of 1 mg/l at pH 4.18 and pH 7.0 and stated that the effect of other pH values corresponds. This means probably, that the growth-pH curve after adding heteroauxin coincides with the dissociation curve of the growth substance. Conformably in the more acid buffer solutions growth was more pronounced. 1 mg/l indole-3-acetic acid caused an increase of about 14 %

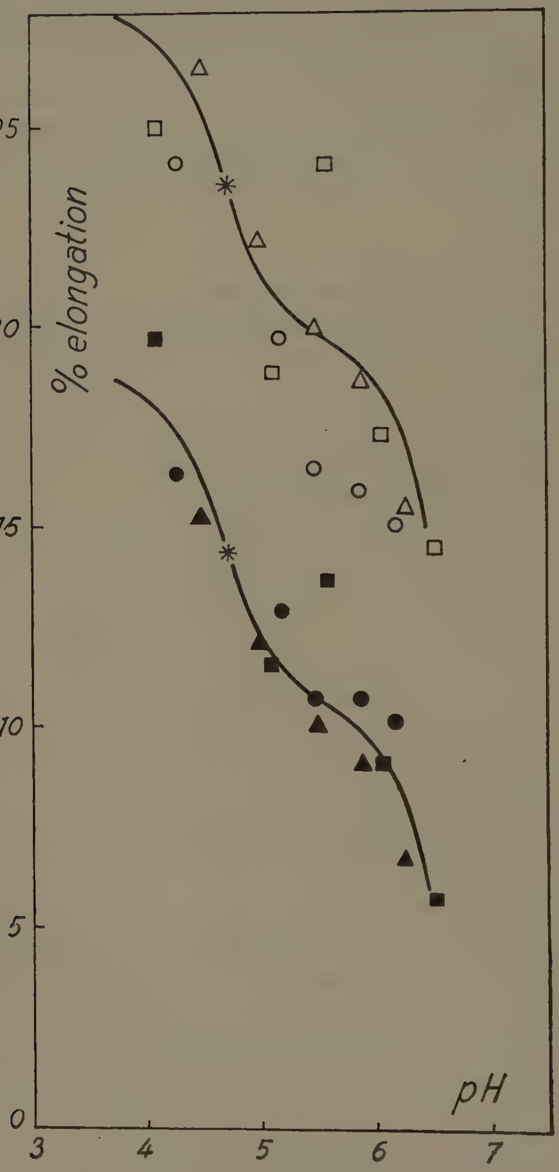

Fig. 2. The influence of pH and ½ % glucose on the growth of *Avena* coleoptile sections. The lower curve refers to growth without glucose, the upper one refers to growth with glucose, both after 24 hours. For the comparability the average values at pH 4.70—4.75 were used as reference points (indicated by asterisks) to calculate the other values.

elongation. In 1940 VAN SANTEN explained the action of the hydrogen ion concentration in terms of the penetration of the undissociated growth substance molecules into the protoplasm. As Miss VAN SANTEN's experiments have not been published in extenso they were repeated on a larger scale.

A concentration of 0.2 mg/l was used instead of 1 mg/l. As the latter concentration is optimal, salt solutions in a certain composition of the medium might permit the I.A.A. molecules to enter the protoplasm to such a degree, that also inside the cell the optimal I.A.A. concentration is reached, whilst in other media this might not be the case. If so, the growth rate at various pH cannot be compared.

The growth was measured 3, 8 and 24 hours after starting the experiment. After the first interval growth proved to be optimal at pH 4.2. When time passed on, this optimum shifted slightly to the alkaline side; after 8 and 24 hours it was moved to pH 4.5 and 4.8 respectively. At the latter moment an extremely low growth rate was observed. At prolongation of the experiment the shape of the curve remained constant. VAN SANTEN

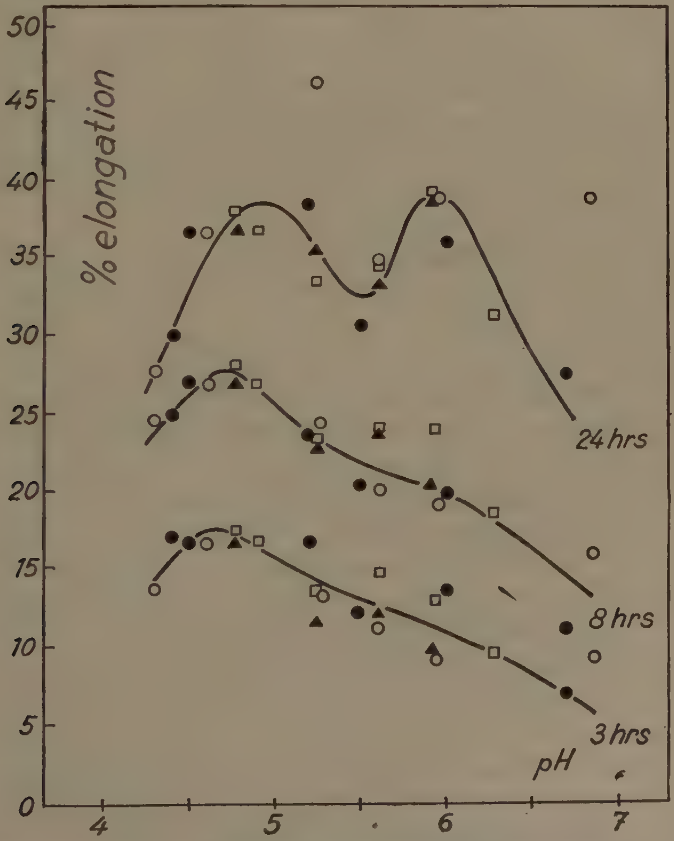

Fig. 3. Relation between the growth of *Avena* coleoptile sections and the external pH after the addition of 0.2 mg/l indole-3-acetic acid. Growth is recorded after 3, 8 and 24 hours in % elongation. For the comparability the average values at pH 4.6—4.9 were taken as reference points to calculate the other values.

(1940) also observed a shift of the optimal pH for growth of roots of *Pisum sativum* to a less acid range. May be the action of the added growth substances is more sensitive to acids than the autonomous growth.

Moreover a remarkable phenomenon was observed. At 10—12 hours from the beginning of the experiment growth between pH 5.6 and 6.2 increased, resulting in a second optimum. At both lower and higher pH values a retardation proved to occur. These results are demonstrated by fig. 3.

It is obvious that these results are in contradiction to the dissociation theory of growth substances.

POHL (1948) did not observe the second optimum, for his experiments were not extended over a 24 hour period but lasted for only 6 hours.

In this respect it is worth mentioning that heteroauxin, though a natural occurring substance, has not definitely been proved to occur in the *Avena* coleoptile. It is generally accepted, that the normal growth substance of these organs consists of auxin-a. Since the latter substance is not available in a pure cristalline state, it is not possible to compare its action with indole-3-acetic acid.

Recently WILDMAN and BONNER (1948) claim to have shown, that the bulk of the growth substance in the *Avena* coleoptile consists of indole-3-acetic acid. In a future paper some experiments will be published, to show that in the *Avena* coleoptile, besides an alkaline stable growth substance, an acid stable growth substance occurs as well. It is highly probable that this substance is identical with auxin-a.

It would be of great importance to compare the action of auxin-a and indole-3-acetic acid, but until we cannot dispose of pure auxin-a, an exact comparison is not possible.

§ 4. *Interaction of pH, growth substance and glucose.*

In the preceding paragraphs experiments were described, carried out with glucose and indole-3-acetic acid added separately. It was shown that both substances promote growth, but glucose acts independent of the pH, whilst the action of indole-3-acetic acid is pH sensitive. From these results it was concluded, that the action of both substances is of a different nature.

We next investigated, whether an interaction of both substances occurs after adding 0.2 mg/l indole-3-acetic acid and ½ % glucose simultaneously. Fig. 4 shows the results: two pH optima arise, both at the same values as those observed with hetero-auxin only (fig. 3).

Nevertheless there are some differences. The growth influenced by I.A.A. at pH 4.8—5.2 and at pH 5.9 is equal, but after the addition of glucose the growth at pH 5.9 is far more pronounced, numbering on an average 78 % at pH 5.9 and 66 % at pH 4.8.

Another difference from the results of § 2 and § 3 is the elongation caused by both substances seperately. At pH 4.8 ½ % glucose causes about 10 % elongation, growth with indole-3-acetic acid amounts to 37 % and growth with both substances together amounts to 66 %.

Hence the influence of glucose on growth is stimulated by the addition of heteroauxin, viz. 10 % and 29 % resp.

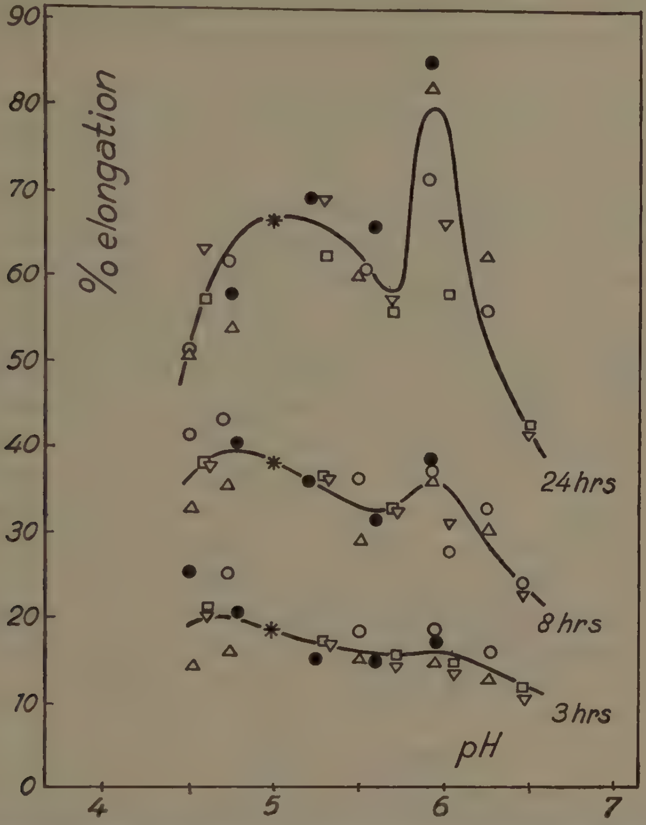

Fig. 4. Relation between the growth of *Avena* coleoptile sections and the external pH after the addition of 0.2 mg/l indole-3-acetic acid and ½ % glucose. Growth is recorded after 3, 8 and 24 hours in % elongation. For the comparability the average values at pH 5.0 were taken as reference points (indicated by asterisks) to calculate the other values.

We may conclude from these experiments that glucose interacts with indole-3-acetic acid, the effect dominating at pH 5.9, but well defined at other pH values as well. The action of glucose may be dual, as it promotes growth also without I.A.A.

COMMONER and THIMANN (1941) supposed, that indole-3-acetic acid exerted an effect on the respiration mechanism. It cannot be decided yet, whether this action holds true and may be the cause of the observed phenomena.

V. *Summary.*

1. From experiments on the pH of the cell contents it is concluded, that only the pH of the outer layers of the protoplasm may be influenced by the medium. The pH of both the inner layers and the vacuole is not to be influenced in this way, although some effect of the salt solutions on the

inner layers may not be excluded. Probably a new ion equilibrium between the medium and the protoplasm is the cause of the pH effect.

2. Experiments on the growth of *Avena* coleoptile sections confirmed the results of VAN SANTEN (1940) with regard to the effect of the pH on growth. Optimal growth occurs at pH 4.2.

3. Glucose added to the medium causes a considerable growth promotion; the influence of the pH on growth is unaffected. Obviously two processes are involved in growth.

4. Indole-3-acetic acid promotes growth dependent of the pH. Optimal growth occurs at pH 4.8—5.2 instead of pH 4.2. Apart from this optimum a second one is observed between pH 5.6 and pH 6.0, caused by an increase of the growth rate after 10—12 hours in this pH range.

5. The combined action of heteroauxin and glucose also results in two pH optima, the optimum at pH 5.9 being far more pronounced than without glucose. At other pH values a stimulation of the growth also is observed. These results indicate that the two substances interfere.

Acknowledgements.

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Zoology. — *The Identity of Penaeus monodon Fabr.* By L. B. HOLTHUIS. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of October 29, 1949.)

The genus *Penaeus* Fabr. from a commercial point of view undoubtedly is the most important group of prawns. It occurs in the tropical and subtropical waters of the world and is used for food throughout its range of distribution. To make clear the economic importance of the genus, one only has to mention that the value of the catch of *Penaeus setiferus* (L.) along the South Atlantic and Gulf coast of the United States of America, in 1943 alone, amounted to 15 million dollars (cf. ANONYMUS, 1945, p. 91). In the Mediterranean it is *Penaeus kerathurus* (Forssk.) (the Langostino of the Spanish and Caramote of the French) which is fished and sold for food, while in E. Australia *Penaeus plebejus* Hess and *Penaeus esculentus* Haswell are caught in huge quantities. In the rest of the indo-westpacific region too the prawns of the genus *Penaeus* play an important rôle in the commercial fisheries.

At present in India and Indonesia much attention is given to the problem of intensifying the prawn fisheries and developing the prawn industries. These economic studies of the prawns are greatly complicated by the fact that the nomenclature of two of the indo-westpacific species is hopelessly confused. It is the object of the present paper to establish the correct names of the two species involved and to be in this way of some help to those who study the prawn fisheries of the indo-westpacific area.

The two species with which we are concerned here, have been given most frequently the names *Penaeus monodon* Fabricius (1798), *Penaeus semisulcatus* De Haan (1841), and *Penaeus carinatus* Dana (1852), while also the names *Penaeus tahitensis* Heller (1862) and *Penaeus ashiaka* Kishinouye (1900) have been used for them.

In 1906 ALCOCK gave a revision of the Indian species of the genus *Penaeus*. In this revision he includes two species, which he names *Peneus monodon* Fabricius and *Peneus semisulcatus* De Haan. The main difference between these two forms is that in the former the fifth pair of legs bears a small exopod, while this exopod is absent in the latter. Now in 1911 DE MAN, who had examined the type specimens of *Penaeus semisulcatus* De Haan, pointed out that these type specimens belong to the species which bears a small exopod on the last pereiopods and thus is different from ALCOCK's *Peneus semisulcatus*, it probably should be identical with his *monodon*. DE MAN does not use the name *monodon* at all, but gives the name *Penaeus carinatus* Dana to the species which ALCOCK named *P. semisulcatus*. The other species is named by him *Penaeus semisulcatus* De Haan. DE MAN further remarks that the species to which has been given the name *P. carinatus* Dana eventually may get the name *Penaeus monodon* Fabr., when the identity of *P. monodon* of Alcock and *P. semisulcatus* De Haan is established. Then KEMP (1915, p. 317) proves the identity of *Peneus monodon* of ALCOCK and *P. semisulcatus* De Haan. KEMP prefers, however, not to use the name *monodon* at all because as "Alcock's application of *monodon* is shown to be incorrect, I do not think it can safely be used for any other species". Most of the later authors as PARISI (1919), SCHMITT (1926), and BALSS (1933) follow KEMP in using the names *P. semisulcatus* and *P. carinatus*.

The first question occurring to us is: Is KEMP right in suppressing the name *monodon* in this case? In my opinion this question must be answered in the negative. FABRICIUS, when establishing his new genus *Penaeus* 1) in 1798 included in it three species: *P. monodon, P. monoceros* and *P. planicornis*. Of these only *P. monodon* at present is maintained in the genus *Penaeus, P. monoceros* is considered to form part of the genus *Penaeopsis* Bate, while *P. planicornis* is a species incerta, which certainly does not belong to the genus *Penaeus* (cf. ALCOCK, 1906, p. 55). *Penaeus monodon* furthermore was made the type of the genus *Penaeus* by LATREILLE (1810, p. 422). As thus *Penaeus monodon* Fabr. is the type species of the genus *Penaeus* and, moreover, is the only species of the genus *Penaeus* in the sense of modern authors which is included by FABRICIUS in the original genus, it would be extremely unpleasant to have this species made a species incerta. In my opinion it seems best to attach the trivial name *monodon* to that species which proves to be most appropriate to bear it; this procedure very often is followed with species inadequately described by LINNAEUS, FABRICIUS and others. In the present case it is rather easy, since *Penaeus monodon* can only be one of two known species, FABRICIUS's description excluding the possibility that it is identical with a species other than the forms named *Penaeus semisulcatus* and *P. carinatus* by DE MAN (1911) and KEMP (1915).

The second question with which we have to occupy ourselves is to which of the two species the name *monodon* should be given. In the first place we have to try and find in the original material and the original description of *Penaeus monodon* indications for applying that name to one of the two species. FABRICIUS's material of the species was collected by DAGOBERT CARL DE DALDORFF, a lieutenant in the Danish East India Company at Tranquebar on the Coromandel Coast of India. According to HORN & KAHLE (1935—1937, p. 50) the DALDORFF collections partly are preserved in the Zoological Museum at Copenhagen, Denmark, and partly in the Zoological Museum of the University at Kiel, Germany. The late Dr. K. H. STEPHENSEN of the Zoological Museum at Copenhagen informed me by letter that his Museum only possesses Insect and Brachyuran types of

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1) The name *Peneus* Weber (1795) is not valid. WEBER namely establishes this genus for the species *P. monodon, P. monocerus,* and *P. planicornis,* of which he only gives the names and which for the first time are described by FABRICIUS in 1798.

FABRICIUS. Professor Dr. WOLF HERRE, director of the Zoological Museum at Kiel, was so kind to send me a list of the Fabrician Decapods in his Institution, which fortunately all have come safely through World War II. Though the list contains two Caridean names (*Palaemon carcinus*, which of course is not a type as the species was first described by LINNAEUS as *Cancer Carcinus*, and *Palaemon quadricornis*, which may be an error for *Palinurus quadricornis*), no Penaeidea are included in it. The Penaeid types of FABRICIUS thus are no longer extant and can not be used for solving the puzzle of the identity of *Penaeus monodon*.

FABRICIUS's description of *Penaeus monodon* runs as follows:

"P[enaeus]. rostro porrecto adscendente supra serrato subtus tridentato.

Habitat in Oceano Indico edulis Dom. Daldorff.

Corpus maiusculum, variegatum. Thorax laeuis dentibus anticis utrinque duobus. Rostrum porrectum, adscendens, supra serratum, subtus tridentatum. Chelae sex filiformes posticis longioribus."

The description of the animal itself makes it possible only to decide that the species is either *P. semisulcatus* De Haan or *P. carinatus* Dana, to use DE MAN's (1911) and KEMP's (1915) nomenclature. As already stated above DALDORFF lived at Tranquebar and it is most probable that his specimen(s) originate(s) from there. The fact that FABRICIUS states that the species is „edulis" is a strong indication for the possibility that DALDORFF obtained this material from the fish market or from fishermen. Now ALCOCK (1906, p. 11) in his revision of the Indian *Penaeus* species states that his *Penaeus semisulcatus*, which is the *Penaeus carinatus* of later authors, "is the commonest salt-water prawn of the Calcutta market", and CHOPRA (1943, p. 2) in his paper on the prawn fisheries of India extensively deals with *Penaeus carinatus* as a very important food prawn, while no mention is made of *P. semisulcatus*. This evidence thus makes it very probable that *Penaeus monodon* Fabr. is identical with *Penaeus carinatus* of modern authors.

In the second place we have to take into account the opinion of "the first reviser", i.e., in the present case, the first author who recognises and treats the two forms as distinct species. This author undoubtedly is DE HAAN (1849, p. 190), who gives a key to the species of *Penaeus* known at that moment. In this key he separates *Penaeus semisulcatus* from *P. monodon* by the fact that in the former species "sulcus a basi rostri ad marginem posteriorem thoracis", while in the latter "sulcus inter basin rostri et marginem posticum thoracis nullus". Now in *P. semisulcatus* the post-rostral carina is always distinctly sulcate, while in *P. carinatus* the "post-rostral carina [is] usually more or less sulcate" (SCHMITT, 1926, p. 359). The character of the presence or absence of the groove is not very constant in *P. carinatus*, but the fact that it is absent or vague in DE HAAN's *P. monodon* already proves that that species can not be identical with *P. semisulcatus* De Haan and thus only can be *P. carinatus* since the shape of the rostrum as described by DE HAAN excludes the possibility of being identical with other Indian Penaeids. The decision taken by the first reviser thus also confirms that *Penaeus carinatus* and *P. monodon* must be considered synonyms.

In the third place we have to consider the practical advantages and disadvantages of applying the name *monodon* to either *P. carinatus* or *P. semisulcatus*. When we give the name *monodon* to the species named *Penaeus carinatus* by DE MAN and KEMP, the other species automatically keeps the name *semisulcatus*. *Penaeus semisulcatus* De Haan is a well established species of which a good original figure and description were given and of which the types still are extant, so that there is not the slightest doubt as to the identity of the species. The only disadvantage as I see it is that the names *semisulcatus* and *monodon* then are used exactly in the opposite sense as it was done by ALCOCK (1906) in his revision of the Indian *Penaeus* prawns. ALCOCK is the first to give really good and reliable characters to separate the two forms; before 1906 the name *monodon* has been used for both species. Five years after the publication of ALCOCK's paper DE MAN already pointed to the fact that ALCOCK had used the name *semisulcatus* in an incorrect sense. Since that time ALCOCK's nomenclature for the two species has been abandoned, though his revision still is one of the most important papers on the indo-westpacific Penaeids. The restoration of the name *monodon* for ALCOCK's *Peneus semisulcatus* probably will cause as little confusion as the substitution of the name *semisulcatus* for his *monodon* did. When we take the other course and use the name *monodon* for *P. semisulcatus* De Haan, the name *monodon* as used by ALCOCK will remain, but his *semisulcatus* nevertheless has to disappear. For the latter species most modern authors use the name *Penaeus carinatus* Dana, of which, as KEMP (1915, p. 317) already pointed out, it is doubtful whether it really is identical with the species named *semisulcatus* by ALCOCK. DANA's description and figure are not sufficient to establish the identity of the two forms beyond doubt. Though we may accept with the larger part of the modern authors the identity of *Penaeus carinatus* Dana with the form named *P. semisulcatus* by ALCOCK, we still may not use DANA's name, since the name *Penaeus carinatus* Dana (1852) is preoccupied by that of *Penaeus carinatus* Otto (1821). The only other name which remains for the species then is *Penaeus tahitensis* Heller (1862). PESTA (1912) examined the type specimens of HELLER's species, which are preserved in the Vienna Museum, and came to the conclusion that they are very probably identical with *Penaeus carinatus*, but as the specimens are damaged no full certainty could be obtained in this respect. When we thus should keep the name *Penaeus monodon* Fabr. for *P. semisulcatus* De Haan, the other species has to have the name *Penaeus tahitensis* Heller or has to receive a new one.

Taking all this evidence into account, I am convinced that the only correct thing to do is to give the name *Penaeus monodon* Fabr. to the species indicated with the name *Penaeus carinatus* Dana by DE MAN

(1911), KEMP (1915) and many subsequent authors. The other species has to bear the name *Penaeus semisulcatus* De Haan.

As the real types of *Penaeus monodon* are lost, I have indicated a specimen of this species in the collection of the Leiden Museum as the neotype of *Penaeus monodon* Fabr.

A short account of the material of the two species preserved in the Rijksmuseum van Natuurlijke Historie at Leiden is given here.

Penaeus monodon Fabricius, 1798

Synonyms: *Penaeus carinatus* Dana, 1852 (non Otto, 1821); *Penaeus tahitensis* Heller, 1862; *Peneus semisulcatus* Alcock, 1906 (non De Haan, 1841).

Last pereiopod without exopod. Rostral carina with the sulcus more or less distinct. Carina and groove running upwards from the hepatic spine only feebly indicated.

The collection of the Leiden Museum contains the following specimens:

Pulu Weh, off N. Sumatra; 1910, January 1917, August 1925; leg. P. BUITENDIJK. — 4 ♂♂ 71—156 mm, 2 ♀♀ 108 and 114 mm.
Atjeh (= Atchin); 1879; leg. WALRAVEN. — 1 ♂ 180 mm.
Belawan Deli, N.E. Sumatra; May 1929; leg. P. BUITENDIJK. — 1 ♀ 71 mm.
Noordpoel, Verlaten Island, Strait Sunda; December 1933; leg. K. W. DAMMERMAN. — 2 ♂♂ 145 and 147 mm.
Java Sea; 1906; leg. P. BUITENDIJK. — 1 ♂ 121 mm, 2 ♀♀ 92 and 220 mm.
West Java; 1914; leg. J. F. VAN BEMMELEN. — 3 ♂♂ 195—227 mm.
Batavia; leg. P. BLEEKER. — 1 ♂ 200 mm.
Batavia; 1896; leg. A. G. VORDERMAN. — 1 ♀ 158 mm.
Bay of Batavia; January 1908, August 1908, June 1924; leg. P. BUITENDIJK. — 3 ♂♂ 116—200 mm, 3 ♀♀ 76—210 mm.
Bay of Batavia; July 1938; leg. F. P. KOUMANS. — 2 ♂♂ 85 and 96 mm.
Tandjong Priok, harbour of Batavia; November 1926, August 1927; leg. P. BUITENDIJK. — 1 ♂ 92 mm, 2 ♀♀ 84 and 90 mm.
Mouth of river W. of Tandjong Priok; July 1911; leg. P. BUITENDIJK. — 1 ♀ 137 mm.
Coast near Tandjong Priok; 1906; leg. P. BUITENDIJK. — 1 ♀ 190 mm.
Near Tjiliwung River, near Batavia; 1906; leg. P. BUITENDIJK. — 1 ♀ 200 mm.
Kampong Makasar near Batavia; leg. Mr. GROEN. — 1 ♀ 195 mm.
Indramaju, north coast of W. Java; September 1924; leg. P. BUITENDIJK. — 3 ♂♂ 57—60 mm, 1 ♀ 77 mm.
Tegal, north coast of Central Java; November 1927; leg. P. BUITENDIJK. — 1 ♂ 90 mm.
Off Semarang, north coast of Central Java; December 1910, March 1912; leg. P. BUITENDIJK. — 3 ♂♂ 98—200 mm, 1 ♀ 136 mm.
Surabaja, E. Java; November 1926, February 1927, November 1927, June 1930; leg. P. BUITENDIJK. — 3 ♂♂ 55—90 mm, 3 ♀♀ 53—136 mm.
Pasuruan, north coast of E. Java; May 1929; leg. P. BUITENDIJK. — 2 ♀♀ 46 and 52 mm.
Probolinggo, north coast of E. Java; August 1, 1924; leg. P. BUITENDIJK. — 1 ♀ 65 mm.
Madura; July 1920, July 1928, March 1930; leg. P. BUITENDIJK. — 1♂ 58 mm, 2 ♀♀ 88 and 116 mm.
Ondolean, E. Celebes; December 18, 1904. — 1 ♀ 130 mm.
Makassar, S.W. Celebes; leg. D. M. PILLER. — 3♀♀ 125—210 mm.
Moluccas; 1895; leg. W. A. MOREAUX. — 1 ♂ 200 mm, 3 ♀♀ 200—220 mm.
Japan. — 1 ♂ 92 mm.

South Seas; 1887; Museum Godeffroy. — 1 ♀ 128 mm.
Locality unknown. — 2 ♂♂ 78 and 162 mm, 1 ♀ 108 mm.

Type. A male of 200 mm length from the Bay of Batavia (June 1924, leg. P. BUITENDIJK) has been chosen as the neotype of this species.

Penaeus semisulcatus De Haan

Synonyms: *Penaeus ashiaka* Kishinouye, 1900; *Peneus monodon* Alcock, 1906 (non Fabricius, 1798).

Last pereiopod provided with a small exopod. Rostral carina with the sulcus always distinct. Carina and groove running upwards from hepatic spine much stronger and sharper than in *Penaeus monodon*.

The collection of the Leiden Museum possesses the following material of this species:

Jidda, Red Sea; 1880; leg. J. A. KRUYT. — 3 ♂♂ 53—95 mm, 2 ♀♀ 90 and 123 mm.
Japan; types. — 1 ♂ 170 mm, 2 ♀♀ 170 and 205 mm.
Japan. — 1 ♂ 155 mm, 1 ♀ 190 mm.
Philippines; 1893; leg. A. VAN DER VALK. — 1 ♂ 138 mm.
Pulu Weh, off N. Sumatra; 1910, January 1913, January 1927, February 1927, April 1928; leg. P. BUITENDIJK. — 4 ♂♂ 62—88 mm, 5 ♀♀ 65—130 mm.
Belawan Deli, N.E. Sumatra; September 1929; leg. P. BUITENDIJK. — 1 ♂ 70 mm.
Buitenzorg [2]); 1909; leg. H. VAN DER WEELE. — 1 ♂ 78 mm.
Batavia; leg. P. BLEEKER. — 1 ♀ 140 mm.
Bay of Batavia; January 1908, August 1926; leg. P. BUITENDIJK. — 1 ♂ 65 mm, 1 ♀ 72 mm.
Mouth of river W. of Tandjong Priok; July 1911; leg. P. BUITENDIJK. — 1 ♀ 111 mm.
Near Tjiliwung River, near Batavia; 1906; leg. P. BUITENDIJK. — 1 ♀ 85 mm.
Cheribon, north coast of W. Java; July 1926; leg. P. BUITENDIJK. — 1 ♀ 63 mm.
Off Semarang, north coast of Central Java; October 1912; leg. P. BUITENDIJK. — 1 ♀ 62 mm.
Makassar, S.W. Celebes; 1888—1889; leg. M. WEBER. — 1 ♂ 80 mm.
Amboina, Moluccas; leg. D. J. HOEDT. — 1 ♀ 105 mm.
Moluccas; 1895; W. A. MOREAUX. — 2 ♂♂ 59 and 63 mm.
Locality unknown. — 1 ♀ 165 mm.

Type. As indicated above there are three cotypes, one male and two females, originating from Japan. It is not certain whether the two other specimens from Japan also are types.

The species *Penaeus semisulcatus* De Haan generally is considered to date from 1849; this is incorrect, however, since plate 46 of the Crustacean volume of the Fauna Japonica, which contains the figure and the name of *Penaeus semisulcatus*, was issued as early as 1841. The text describing the species was published in 1849.

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[2]) This locality probably is not correct, as it lies far in the interior of Java.

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